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Computation of Lyapunov functions for nonlinear discrete systems by linear programming

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Given a nonautonomous discrete system with an equilibrium at the origin and a hypercube $\mathcal{D}$ containing the origin, we state a linear programming problem, of which any feasible solution parameterizes a continuous and piecewise affine (CPA) Lyapunov function $V : \mathcal{D} \to \mathbb{R}$ for the system. The linear programming problem depends on a triangulation of the hypercube. We prove that if the equilibrium at the origin is exponentially stable, the hypercube is a subset of its basin of attraction, and the triangulation fulfills certain properties, then such a linear programming problem possesses a feasible solution. We suggest an algorithm that generates such linear programming problems for a system, using more and more refined triangulations of the hypercube. In each step the algorithm checks the feasibility of the linear programming problem. This results in an algorithm that is always able to compute a Lyapunov function for a discrete system with an exponentially stable equilibrium. The domain of the Lyapunov function is only limited by the size of the equilibrium's domain of attraction. The system is assumed to have a C^2 right-hand side, but is otherwise arbitrary. Especially, it is not assumed to be of any specific algebraic type like linear, piecewise affine, etc. Our approach is a non-trivial adaption of the CPA method to compute Lyapunov functions for continuous systems to discrete systems.

1. Introduction

Consider the discrete dynamical system with an equilibrium at the origin:

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k), \text{where } \mathbf{g} \in C^2(\mathbb{R}^n, \mathbb{R}^n) \text{ and } \mathbf{g}(\mathbf{0}) = \mathbf{0}. \qquad (1)$$

Define the mapping $\mathbf{g}^{\circ m} : \mathbb{R}^n \to \mathbb{R}^n$ for all $m \in \mathbb{N}_0$ by induction through $\mathbf{g}^{\circ 0}(\mathbf{x}) := \mathbf{x}$ and $\mathbf{g}^{\circ (m+1)}(\mathbf{x}) := \mathbf{g}(\mathbf{g}^{\circ m}(\mathbf{x}))$ for all $\mathbf{x} \in \mathbb{R}^n$. The origin is said to be an *exponentially stable equilibrium* of the system (1) if there exist constants $\delta, M > 0$ and $0 < \mu < 1$ such that $\|\mathbf{g}^{\circ m}(\mathbf{x})\| \le \mu^m M \|\mathbf{x}\|$ for all $\|\mathbf{x}\| < \delta$ and all $m \in \mathbb{N}_0$. The set $\mathcal{A} := \{\mathbf{x} \in \mathbb{R}^n : \limsup_{m\to+\infty} \|\mathbf{g}^{\circ m}(\mathbf{x})\| = 0\}$ is called its basin of attraction.

The stability of the equilibrium can be characterized by so-called Lyapunov functions, i.e. continuous functionals on the state-space decreasing along the system trajectories and with a minimum at the equilibrium. Further, Lyapunov functions additionally deliver a lower bound on the basin of attraction. For linear systems, i.e. $\mathbf{g}(\mathbf{x}) = A\mathbf{x}$ for an $A \in \mathbb{R}^{n\times n}$, the origin is an exponentially stable equilibrium of the system, if and only if all eigenvalues λ of A fulfill $|\lambda| < 1$. In this case a quadratic Lyapunov function can be

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constructed for the system by standard methods that ensure $\mathcal{A} = \mathbb{R}^n$ and the system is said to be globally stable, cf. e.g. Lemma 5.7.19 in [38].

If $\mathbf{g}$ is nonlinear, then the classical approach is to consider the linearized system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where $A := D\mathbf{g}(\mathbf{0})$ is the Jacobian matrix of $\mathbf{g}$ at the origin. If the origin is an exponentially stable equilibrium of the linearized system the same holds true for the nonlinear system. However, in this case a quadratic Lyapunov function for the linear system is only a Lyapunov function for the nonlinear system in some local neighbourhood of the origin. Thus, in most cases, it gives a very conservative lower bound on the basin of attraction for the nonlinear system. This is unfortunate, because the size of the basin of attraction is often of great importance. For example in engineering, the system (1) is often a description of some machinery that has to be close to the equilibrium to work as intended. Local stability of the equilibrium translates into "the system can withstand all small enough perturbations" and this property is obviously a necessity if the machinery is to be of any use. However, this property is clearly not sufficient and the robustness of the machinery, i.e. how large perturbations it can withstand, is of central importance. In social sciences or economics, for example, where models and parameters are inheritably subject to considerable uncertainty, the robustness of an equilibrium is of even greater importance.

In such cases and many more, a Lyapunov function for the system, defined on a not merely local neighbourhood of an equilibrium, but with a domain that extends over a reasonable subset of the basin of attraction, gives useful and concrete information on the robustness of an equilibrium. Such Lyapunov functions are, however, much more difficult to construct than the local ones. For some general discussion on the stability of equilibrium points of discrete systems and Lyapunov functions see e.g. chapter 5 in [38] or chapter 5 in [1] and for a more advanced discussion on Lyapunov functions for discrete systems see [20]. For references to Lyapunov stability theory for differential inclusions, a generalization to discrete systems, see the references given in Section 6, where we discuss further research.

Numerical methods to compute Lyapunov functions for nonlinear discrete systems have, for example, been presented in [11, 12], where collocation is used to solve numerically a discrete analog to Zubov's partial differential equation [41] using radial basis functions [8, 40] and in [4, 23], where graph algorithms are used to compute complete Lyapunov functions [9, 35]. For nonlinear systems with a certain structure there are many more approaches in the literature. To name a few, in [34] the parameterization of piecewise-affine Lyapunov functions for linear discrete systems with saturating controls is discussed, [30] is concerned with the computation of Lyapunov functions for (possibly discontinuous) piecewise-affine systems, and in [10] linear matrix inequalities are used to compute piecewise quadratic Lyapunov functions for discrete piecewise-affine systems.

In this paper we adapt the continuous and piecewise-affine (CPA) method to compute Lyapunov functions for continuous systems, first presented in [21, 22] and in a more refined form delivering true Lyapunov functions in [32, 33], to discrete systems. Originally the CPA method for continuous systems was only guaranteed to compute

Lyapunov functions for systems with an exponentially stable [17] or an asymptotically stable [18] equilibrium, if an arbitrary small neighbourhood of the equilibrium was cut out from the domain. In [13–16] this restriction could be removed by introducing a fan-like triangulation near the equilibrium. A similar approach is used for the discrete CPA method in this paper. The non-locality of discrete systems, however, implies that a fundamentally different methodology must be used. The CPA method for continuous systems has been extended to nonautonomous switched systems [19] and to autonomous differential inclusions [2, 3]. The CPA method for discrete systems can, at least with some limitation, be extended to difference inclusions and we discuss this in Section 6. The details of this extension would, however, go beyond the scope of this paper and are a matter of ongoing research.

In this paper, we state in Definition 2.9 a linear programming feasibility problem with the property, that a solution to the problem parameterizes a Lyapunov function for the system, cf. Theorem 2.11. The domain of the Lyapunov function is only limited by the size of the equilibrium's basin of attraction and not by artificial bounds due to the approach as in the classical approach. The exponential stability of an equilibrium of the system (1) is equivalent to the existence of a certain Lyapunov function for the system as shown in Lemma 4.1 and we use this in Theorem 4.2 to prove that the feasibility problem always possesses a solution if the parameters of the problem are chosen in a certain way. Because there are algorithms, e.g. the simplex algorithm, that always find a feasible solution to a linear programming problem if one exists, and because we can adequately scan the parameter space algorithmically, cf. Definition 3.1, this delivers an algorithm that is always able to compute a Lyapunov function, of which the domain is only limited by the basin of attraction, for a system of the form (1) possessing an exponentially stable equilibrium.

The structure of the paper is as follows: In Section 2 we define the Lyapunov functions and the triangulations we will be using and then we state our linear programming problem in Definition 2.9. Then, in Theorem 2.11, we prove that a feasible solution to the linear programming problem parameterizes a CPA Lyapunov function for the system. In Section 3 we deliver an algorithm in Definition 3.1 that systematically generates linear programming problems as in Definition 2.9. In Section 4 we prove the existence of a certain Lyapunov function for systems with an exponentially stable equilibrium in Lemma 4.1 and then use it in Theorem 4.2 to prove that the algorithm from Definition 3.1 will deliver a feasible linear programming problem for any such system. Thus, we can always compute a CPA Lyapunov function for a system with an exponentially stable equilibrium. In Section 5 we give an example of our approach to compute CPA Lyapunov functions and in Section 6 we give some concluding remarks and ideas for future research.

Notations

For a vector $\mathbf{x} \in \mathbb{R}^n$ we write x_i or $(\mathbf{x})_i$ for its i-th component. For $\mathbf{x} \in \mathbb{R}^n$ and $p \geq 1$ we define $\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ the norm. We also define $\|\mathbf{x}\| = \infty$

max where $i\in\{p1-,21,...,n+\}q-|x1i|= 1$. We will repeatedly use the Hölder inequality, and the norm equivalence relations $|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|_p\|\mathbf{y}\|_q$,

$\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq n^{q^{-1}-p^{-1}}\|\mathbf{x}\|_p$ for $+\infty \geq p > q \geq 1$ and $\mathbf{x} \in \mathbb{R}^n$.

The *induced matrix norm* $\|\cdot\|_p$ is defined by $\|A\|_p = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p$. Clearly $\|A\mathbf{x}\|_p \leq \|A\|_p\|\mathbf{x}\|_p$. For a symmetric matrix $P \in \mathbb{R}^{n\times n}$ we denote by $\lambda_{\min}^P$ and $\lambda_{\max}^P$ the minimal and maximal eigenvalue of A, respectively. Further, if P is additionally positive definite, i.e. its eigenvalues are all strictly larger than zero, we define the energetic norm $\|\mathbf{x}\|_P := \sqrt{\mathbf{x}^T P\mathbf{x}}$. The estimate $\sqrt{\lambda_{\min}^P}\,\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_P \leq \sqrt{\lambda_{\max}^P}\,\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$ follows immediately from this definition.

Let $(\mathbf{x}_0,\mathbf{x}_1,...,\mathbf{x}_m)$ be an ordered $(m + 1)$–tuple of vectors in $\mathbb{R}^n$. The set of all convex combinations of these vectors is denoted by $\mathrm{co}(\mathbf{x}_0,\mathbf{x}_1,...,\mathbf{x}_m) := \left\{\sum_{i=0}^m \lambda_i\mathbf{x}_i : 0 \leq \lambda_i \leq 1, \sum_{i=0}^m \lambda_i = 1\right\}$. The vectors $(\mathbf{x}_0,\mathbf{x}_1,...,\mathbf{x}_m)$ are called *affinely independent* if $\sum_{i=1}^m \lambda_i(\mathbf{x}_i - \mathbf{x}_0) = \mathbf{0}$ implies $\lambda_i = 0$ for all $i = 1,...,m$. If $(\mathbf{x}_0,\mathbf{x}_1,...,\mathbf{x}_m)$ are affinely independent, then the set $\mathrm{co}(\mathbf{x}_0,\mathbf{x}_1,...,\mathbf{x}_m)$ is called an m-simplex and the vectors $\mathbf{x}_0,\mathbf{x}_1,...,\mathbf{x}_m$ are said to be its vertices.

An inequality such as $\mathbf{x} \leq \mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are vectors, is always to be understood componentwise, i.e. $x_i \leq y_i$ for all i.

The set of m-times continuously differentiable functions from an open set O to a set P is denoted by $C^m(\mathrm{O},\mathrm{P})$. We denote the closure of a set D by $\overline{\mathrm{D}}$, its interior by D°, and its boundary by $\partial\mathrm{D} := \overline{\mathrm{D}} \setminus \mathrm{D}^\circ$. Finally, B_δ is defined as the open $\|\cdot\|_2$-ball with center $\mathbf{0}$ and radius δ, i.e. $\mathrm{B}_\delta := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < \delta\}$.

Remark 1. It is unusual to define a simplex as the convex combination of the vectors of an ordered tuple, because the resulting set is obviously independent of the particular order of the vectors. For our purposes their order is, however, important and this definition has several advantages, cf. Definition 2.7 and Remark 9.

2.The linear programming problem

In this paper we are interested in exponentially stable equilibria, i.e. the moduli of the eigenvalues of the Jacobian of $\mathbf{g}$ from (1) at the equilibrium at the origin are all strictly less than one. We will show that if the origin is an exponentially stable equilibrium of (1), then a CPA Lyapunov function can be computed algorithmically by using linear programming. Because we are only interested in exponentially stable equilibria at the origin we only need to consider a specific type of Lyapunov function that characterizes this kind of stability. Further, it is advantageous to define the set N of those neighborhoods of the origin that we will repeatedly use in this paper. This is done in the next two definitions.

Definition 2.1.Denote by N the set of all subsets $D \subset \mathbb{R}^n$ that fulfill:

i) D is compact.

ii) The interior D° of D is a connected open neighborhood of the origin.

iii) $D = D^\circ$.

A Lyapunov function for a system is a continuous function $V : D \to \mathbb{R}$, with a local minimum at the equilibrium at the origin, which is decreasing along system trajectories, i.e. $V(\mathbf{g}(\mathbf{x})) < V(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{0}$. Because the dynamics of a discrete system are nonlocal, i.e. $\mathbf{g}(\mathbf{x})$ is not necessarily close to $\mathbf{x}$, the property "decreasing along system trajectories" needs some additional consideration compared to the continuous case.

One must either assume, that D is forward invariant or, more practically, restrict the demand $V(\mathbf{g}(\mathbf{x})) < V(\mathbf{x})$ to all $\mathbf{x}$ in a subset O of D, such that $\mathbf{x} \in O$ implies $\mathbf{g}(\mathbf{x}) \in D$. We follow the second approach. **Definition 2.2**. Let $D,O \in N$, $D \supset O$, and $\|\cdot\|, \|\cdot\|$ be arbitrary norms on $\mathbb{R}^n$.

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A continuous function $V : D \to \mathbb{R}$ is called a Lyapunov function for the system (1) if it fulfills:

i) $\mathbf{g}(\mathbf{x}) \in D$ for all $\mathbf{x} \in O$.

ii) $V(\mathbf{0}) = 0$ and there exist constants $a,b > 0$ such that $a\|\mathbf{x}\| \leq V(\mathbf{x}) \leq b\|\mathbf{x}\|$ for all $\mathbf{x} \in D$.

iii) There exists a constant $c > 0$ such that $V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \leq -c\|\mathbf{x}\|$ for all $\mathbf{x} \in O$.

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Remark 2. Because all norms on $\mathbb{R}^n$ are equivalent and the constants $a,b,c > 0$ are arbitrary, the particular norms $\|\cdot\|$ and $\|\cdot\|$ are not of qualitative but only of

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quantitative importance.

Remark 3. The origin is an exponentially stable equilibrium of the system (1), if and only if it possesses a Lyapunov function in the sense of Definition 2.2. In this case every connected component of a sublevel set $V^{-1}([0,r])$, $r > 0$, that is compact in $\mathcal{O}^\circ$, is a subset of the equilibrium's basin of attraction. Let $\alpha > 0$ be such that $\alpha\|\mathbf{x}\| \le \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$. The sufficiency follows

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directly from the estimate $V(\mathbf{g}(\mathbf{x})) \le (1 - \alpha c/b)V(\mathbf{x})$, which implies $V(\mathbf{x}_k) \le (1 - \alpha c/b)^k V(\mathbf{x}_0)$, and the necessity follows by Lemma 4.1 below. The proposition about the sublevel sets follows, for example, by Theorem 2.2 in [11].

The idea of how to compute a CPA Lyapunov function for the system (1) given a hypercube $\mathcal{D} \in \mathcal{N}$, is to subdivide $\mathcal{D}$ into a set $\mathcal{T} := \{\mathfrak{S}_\nu : \nu = 1,2,\ldots,N\}$ of n-simplices $\mathfrak{S}_\nu$, such that any two simplices in $\mathcal{T}$ intersect in a common face or are disjoint, cf. Definition 2.3. Then we construct a linear programming problem in Definition 2.9, of which every feasible solution parameterizes a CPA function V, i.e. a continuous function that is affine on each simplex in $\mathcal{T}$, cf. Definition 2.4. Then we show in Theorem 2.11 that V is a Lyapunov function for the system in the sense of
Definition 2.2.

Because we cannot use a linear programming problem to check the conditions $a\|\mathbf{x}\| \le V(\mathbf{x}) \le b\|\mathbf{x}\|$ and $V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \le -c\|\mathbf{x}\|$ for more that finitely many $\mathbf{x}$,

*

the essence of the linear programming problem is how to ensure that this holds for all $\mathbf{x} \in \mathcal{D}$ and all $\mathbf{x} \in \mathcal{O} \subset \mathcal{D}$, respectively, by only using a finite number of points $\mathbf{x}$.

We start by defining general triangulations and CPA functions, then we define the triangulations we use in this paper and derive their basic properties.

Definition 2.3 (Triangulation.) Let $\mathcal{T}$ be a collection of n-simplices $\mathfrak{S}_\nu$ in $\mathbb{R}^n$. $\mathcal{T}$

eitheris called a triangulation of the set$\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \emptyset$ or $\mathfrak{S}_\nu$ and $\mathcal{D}\mathfrak{S}_\mu$ intersect in a common face. The latter means,$:= \bigcup_{\mathfrak{S}_\nu \in \mathcal{T}} \mathfrak{S}_\nu$ if for every $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$, $\nu \neq \mu$,

with

$\mathfrak{S}_\nu = \mathrm{co}\left(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\right)$ and $\mathfrak{S}_\mu = \mathrm{co}\left(\mathbf{x}_0^\mu, \mathbf{x}_1^\mu, \ldots, \mathbf{x}_n^\mu\right)$,

that there are permutations α and β of the numbers $0,1,2,\ldots,n$ such that $\mathbf{z}_i := \mathbf{x}_{\alpha(i)}^\nu = \mathbf{x}_{\beta(i)}^\mu, \quad \text{for } i = 0,1,\ldots,k$, where $0 \le k < n$,

and

$$\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \mathrm{co}(\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_k).$$

Note that according to Definition 2.3, two simplices $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ with different indices $\nu \neq \mu$ are different, so every simplex is only counted once.

Definition 2.4 (CPA function.) Let T be a triangulation of a set D ⊂ R^n. Then we can define a continuous, piecewise affine function $P : D \to R$ by fixing its values at the vertices of the simplices of the triangulation T . More exactly, assume that for every vertex **x** of every simplex $S_\nu \in T$ we are given a unique real number $P_{\mathbf{x}}$. In particular, if **x** is a vertex of $S_\nu \in T$ and **y** is a vertex of $S_\mu \in T$ and **x** = **y**, then $P_{\mathbf{x}} = P_{\mathbf{y}}$. Then we can uniquely define a function $P : D \to R$ through:

i) $P(\mathbf{x}) := P_{\mathbf{x}}$ for every vertex **x** of every simplex $S_\nu \in T$. ii) P is affine on every simplex $S_\nu \in T$.

The set of such continuous, piecewise affine functions D → R fulfilling i) and ii) is denoted by CPA[T].

Remark 4. If $P \in$ CPA[T] then for every $S_\nu \in T$ there is a unique vector $\mathbf{a}_\nu \in R^n$ and a unique number $b_\nu \in R$, such that $P(\mathbf{x}) = \mathbf{a}^T_\nu \mathbf{x} + b_\nu$ for all $\mathbf{x} \in S_\nu$. Further, if $\mathbf{x} \in \mathfrak{S}_\nu = \operatorname{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu) \in \mathcal{T}$, then **x** can be written uniquely as a convex combination $\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i^\nu$, $0 \leq \lambda_i \leq 1$ for all $i = 0,1,\ldots,n$, and $\sum_{i=0}^{n} \lambda_i = 1$, of the vertices of S_ν and

$$P(\mathbf{x}) = P\left(\sum_{i=0}^{n} \lambda_i \mathbf{x}_i^\nu\right) = \sum_{i=0}^{n} \lambda_i P(\mathbf{x}_i^\nu) = \sum_{i=0}^{n} \lambda_i P_{\mathbf{x}_i^\nu}.$$

Remark 5. For the construction of our triangulations we use the set S_n of all permutations of the numbers $1,2,\ldots,n$, and the standard orthonormal basis $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n$

one ifof R^n. For a set$i \in$ J and equal to zero ifJ = {1,2,...,}i /, we define the characteristic function∈ J. Further, we use the functions $\mathbf{R}\chi$JJ:(iR)$_n$equal to→ R_n,

defined by

$$\mathbf{R}_J(\mathbf{x}) := \sum_{i=1}^{n} (-1)^{\chi_J(i)} x_i \mathbf{e}_i.$$

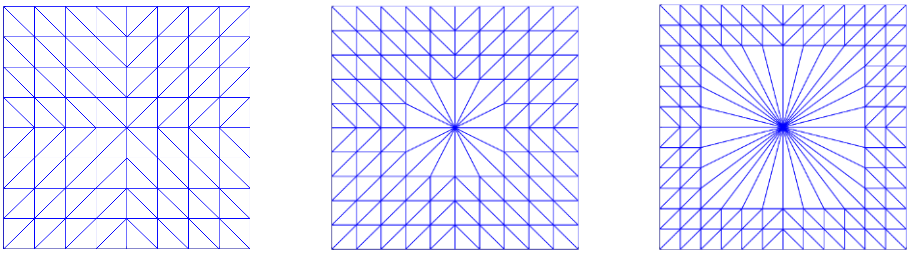
$\mathbf{R}_J(\mathbf{x})$ puts a minus in front of the coordinate x_i of **x** whenever $i \in$ J.

Remark 6. The two parameters b and K of the triangulation $\mathcal{T}_{K,b}^{\text{std}}$, cf. Definition 2.5, refer to the size of the hypercube $[-b,b]^n$ covered by its simplicial fan at the origin and to the fineness of the triangulation, respectively. For schematic pictures of some of these triangulations in 2D see Figure 1. For similar pictures in 3D see Figure 1 in
[15].

Definition 2.5 (Standard triangulations.) We are interested in three general triangulations $\mathcal{T}^{\text{std}}$, $\mathcal{T}_K^{\text{std}}$, and $\mathcal{T}_{K,b}^{\text{std}}$ of $\mathbb{R}^n$.

(1) The triangulation $\mathcal{T}^{\text{std}}$ consists of the simplices

$$\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \operatorname{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$$



(a) $\mathcal{T}^{\text{std}} = \mathcal{T}_{0,1}^{\text{std}}$. (b) $\mathcal{T}_{1,b}^{\text{std}}$. (c) $\mathcal{T}_{2,b}^{\text{std}}$.

Figure 1. Schematic pictures in 2D of some of the triangulations used in this paper.

for all $\mathbf{z} \in \mathbb{N}_0^n$, all $\mathcal{J} \subset \{1,2,\ldots,n\}$, and all $\sigma \in S_n$, where

$$\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} := \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{j=1}^{i} \mathbf{e}_{\sigma(j)}\right) \text{ for } i = 0,1,2,\ldots,n. \quad (2)$$

(2) Choose a $K \in \mathbb{N}_0$ and define the hypercube $\mathcal{H}_K := [-2^K, 2^K]^n$. For every simplex $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} = \operatorname{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right) \in \mathcal{T}^{\text{std}}$, such that $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma} \in \mathcal{H}_K^\circ$ and $\|\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}\|_\infty = 2^K$ for $i = 1,2,\ldots,n$ consider the n-simplex $\mathfrak{S}_{\mathbf{0},\mathbf{z}\mathcal{J}\sigma} := \operatorname{co}\left(\mathbf{0}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$. The set of all such simplices $\mathfrak{S}_{\mathbf{0},\mathbf{z}\mathcal{J}\sigma}$ is denoted by $\mathcal{T}_K^{\text{std}}$. $\mathcal{T}_K^{\text{std}}$ is a triangulation of $\mathcal{H}_K$, cf. Lemma 2.6.

(3) Now choose a constant $b > 0$ and scale the simplices in the triangulation $\mathcal{T}_K^{\text{std}}$ of the hypercube $\mathcal{H}_K$ and the simplices in the triangulation $\mathcal{T}^{\text{std}}$ outside of the open hypercube $\mathcal{H}_K^\circ$ with the mapping $\mathbf{x} \mapsto \rho\mathbf{x}$, where $\rho := 2^{-K}b$. We denote by $\mathcal{T}_{K,b}^{\text{std}}$ the resulting set of n-simplices, i.e.

$$\mathcal{T}_{K,b}^{\text{std}} := \rho\mathcal{T}_K^{\text{std}} \cup \rho\left\{\mathfrak{S} \in \mathcal{T}^{\text{std}} : \mathfrak{S} \cap]-2^K, 2^K[^n = \emptyset\right\}.$$

Remark 7. The triangulations $\mathcal{T}^{\text{std}}$, $\mathcal{T}_K^{\text{std}}$, and $\mathcal{T}_{K,b}^{\text{std}}$ are the same as in [16], but $\mathcal{T}_K^{\text{std}}$ is defined in a more constructive way. This more constructive definition is advantageous for the implementation of the linear programming problem in Definition 2.9. However, we need to prove that $\mathcal{T}_K^{\text{std}}$ is actually a triangulation of $\mathcal{H}_K$. For future use we prove a slightly more general result. The vectors $\mathbf{K}^m$ and $\mathbf{K}^p$ in the following lemma are $(-2^K, -2^K, \ldots, -2^K)^T$ and $(2^K, 2^K, \ldots, 2^K)^T$ respectively for $\mathcal{H}_K$ from Definition 2.5.

Note that the condition (3) in the following lemma is equivalent to assuming that exactly one vertex of the simplex is in $\mathcal{K}^\circ$ and all others are in $\partial\mathcal{K}$, see Remark 8.

Lemma 2.6. *Let $\mathbf{K}^m, \mathbf{K}^p \in \mathbb{Z}^n$ be vectors of negative and positive integers respectively, i.e. $\mathbf{K}^m < \mathbf{0} < \mathbf{K}^p$, and define $\mathcal{K} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{K}^m \leq \mathbf{x} \leq \mathbf{K}^p\}$.*

Let $\mathcal{T}$ denote the set of n-simplices $\mathfrak{S} = \operatorname{co}\left(\mathbf{0}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$, obtained by taking a simplex $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \operatorname{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right) \in \mathcal{T}^{\text{std}}$ cf. (2), such that

$$\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma} \in \mathcal{K}^\circ \text{ and } \{\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\} \subset \partial\mathcal{K}, \tag{3}$$

and replacing the vertex $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}$ by $\mathbf{0}$.

Then $\mathcal{T}$ is a triangulation of $\mathcal{K}$ in the sense of Definition 2.3.

Proof. We start the proof by noting that $(\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma})_{\sigma(1)} = (\mathbf{K}^p)_{\sigma(1)}$ for all $i = 1, \ldots, n$ if $\sigma(1) \notin \mathcal{J}$ and $(\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma})_{\sigma(1)} = (\mathbf{K}^m)_{\sigma(1)}$ for all $i = 1,\ldots,n$ if $\sigma(1) \in \mathcal{J}$. Indeed, this statement follows directly from (2).

Now, we show that the intersection of two different simplices in $\mathcal{T}$ is the convex combination of their common vertices. For this let $\mathfrak{S}_1, \mathfrak{S}_2 \in \mathcal{T}$ be arbitrary.

Then there are $\mathbf{z}, \mathbf{z}^* \in \mathbb{N}_0^n$, $\mathcal{J}, \mathcal{J}^* \subset \{1, 2, \ldots, n\}$, and $\sigma, \sigma^* \in S_n$ such that $\mathfrak{S}^1 = \operatorname{co}\left(\mathbf{0}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$ and $\mathfrak{S}^2 = \operatorname{co}\left(\mathbf{0}, \mathbf{x}_1^{\mathbf{z}^*\mathcal{J}^*\sigma^*}, \mathbf{x}_2^{\mathbf{z}^*\mathcal{J}^*\sigma^*}, \ldots, \mathbf{x}_n^{\mathbf{z}^*\mathcal{J}^*\sigma^*}\right)$. Since $\mathcal{T}^{\text{std}}$ is a triangulation, we have

$$\mathfrak{S}_1 \cap \mathfrak{S}_2 \cap \partial\mathcal{K} = \mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} \cap \mathfrak{S}_{\mathbf{z}^*\mathcal{J}^*\sigma^*} \cap \partial\mathcal{K} = \operatorname{co}(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k),$$

where $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ are the common vertices of $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$ and $\mathfrak{S}_{\mathbf{z}^*\mathcal{J}^*\sigma^*}$ in $\partial\mathcal{K}$. If $0 \leq k < n$, then we have $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \operatorname{co}(\mathbf{0}, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k)$. Indeed, it is clear that $\mathfrak{S}_1 \cap \mathfrak{S}_2 \supset \operatorname{co}(\mathbf{0}, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k)$. On the other hand, let $\mathbf{x} \in \mathfrak{S}_1 \cap \mathfrak{S}_2 \setminus \{\mathbf{0}\}$. As $\mathbf{0}, \mathbf{x} \in \mathcal{K}$ and $\mathcal{K}$ is convex, there is a $\nu \geq 1$ such that $\mathbf{x}_* := \nu\mathbf{x} \in \partial\mathcal{K}$.

We will now show that $\mathbf{x}^* \in \mathfrak{S}_1 \cap \mathfrak{S}_2$. Since $\mathbf{x} \in \mathfrak{S}_1$, we have

$$\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}$$

with $\sum_{i=0}^{n} \lambda_i = 1,\ \lambda_i \geq 0$ and $\lambda_0 = 0$. Then

$$\nu\mathbf{x} = \sum_{i=1}^{n} \nu\lambda_i \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}.$$

We show that $\sum_{i=1}^{n} \nu\lambda_i \leq 1$. Indeed, assuming $\sigma(1) \notin J$ and using the statement at the beginning of the proof, $(\sum_{i=1}^{n} \nu\lambda_i \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma})_{\sigma(1)} = \sum_{i=1}^{n} \nu\lambda_i (\mathbf{K}^p)_{\sigma(1)} \leq (\mathbf{K}^p)_{\sigma(1)}$, since $\nu\mathbf{x} \in K$. A similar argument holds for $\sigma(1) \in J$ and S_2.

Now, since $\mathbf{x}^* \in S_1 \cap S_2 \cap \partial K$, we have

$$\mathbf{x}_* = \nu\mathbf{x} = \sum_{i=1}^{k} \mu_i \mathbf{z}_i$$

with $\sum_{i=1}^{k} \mu_i = 1$ and thus

$$\mathbf{x} = \sum_{i=1}^{k} \frac{\mu_i}{\nu} \mathbf{z}_i + \left(1 - \sum_{i=1}^{k} \frac{\mu_i}{\nu}\right) \mathbf{0}$$

where $1 - \sum_{i=1}^{k} \frac{\mu_i}{\nu} \geq 0$ since $\nu \geq 1$. This shows that $\mathbf{x} \in \mathrm{co}(\mathbf{0}, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k)$.

Case k = n

Now we consider the case $k = n$. We will show that $\mathbf{z} = \mathbf{z}_*$, $J = J_*$, and $\sigma = \sigma_*$, i.e. that we do not obtain the same simplex twice.

By (3), we have $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_0^{\mathbf{z}^*\mathcal{J}^*\sigma^*} \notin \partial\mathcal{K}$ and $\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_i^{\mathbf{z}^*\mathcal{J}^*\sigma^*} \in \partial\mathcal{K}$ for all $i = 1,2,\ldots,n$.

Now consider $\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \mathbf{e}_{\sigma(1)}\right) \in \partial\mathcal{K}$; hence, there is an $n^* \in \{1,2,\ldots,n\}$ such that (i) $(\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma})_{n^*} = (\mathbf{K}^p)_{n^*}$ and $n^* \notin \mathcal{J}$ or (ii) $(\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma})_{n^*} = (\mathbf{K}^m)_{n^*}$ and $n^* \in \mathcal{J}$.

We only consider case (i); case (ii) can be dealt with similarly. Since $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma} \notin \partial\mathcal{K}$, we have $(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma})_{n^*} < (\mathbf{K}^p)_{n^*}$; in particular $\sigma(1) = n_*$. By assumption there is an $i^* \in \{1,2,\ldots,n\}$ such that

$$\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{x}_{i^*}^{\mathbf{z}^*\mathcal{J}^*\sigma^*} = \mathbf{R}^{\mathcal{J}^*}\left(\mathbf{z}^* + \sum_{j=1}^{i^*} \mathbf{e}_{\sigma^*(j)}\right). \quad (4)$$

This implies $n^* \notin J_*$, since $(\mathbf{x}_{i^*}^{\mathbf{z}^*\mathcal{J}^*\sigma^*})_{n^*} = (\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma})_{n^*} > 0$.

There are three cases, either (i) $\sigma(1) = \sigma_*(1)$, (ii) $\sigma(1) \in \{\sigma_*(2), \sigma_*(3), \ldots, \sigma_*(i_*)\}$ or (iii) $\sigma(1) \in \{\sigma_*(i_* + 1), \sigma_*(i_* + 2), \ldots, \sigma_*(n)\}$. We need to exclude cases (ii) and (iii).

In case (iii), the $\sigma(1) = n_*$-th component of

$$\mathbf{R}^{\mathcal{J}^*}\left(\mathbf{z}^* + \sum_{j=1}^{n} \mathbf{e}_{\sigma^*(j)}\right)$$

is equal to $(\mathbf{K}^p)_{n_*} + 1$, i.e. the point is not in ∂K – a contradiction.

In case (ii), let $\sigma(1) = \sigma_*(j_*)$ with $2 \le j^* \le i_*$, then the $\sigma(1) = n_*$-th component of

$$\mathbf{x}^{\mathbf{z}^*\mathcal{J}^*\sigma^*}_{j^*-1} = \mathbf{R}^{\mathcal{J}^*}\left(\mathbf{z}^* + \sum_{j=1}^{j^*-1} \mathbf{e}_{\sigma^*(j)}\right)$$

is equal to $(\mathbf{K}^p)_{n_*} - 1$. The point is in ∂K (as it is not $\mathbf{x}_0^{\mathbf{z}^*\mathcal{J}^*\sigma^*}$), hence, there is an $m_* \neq n_*$ such that (i) $\left(\mathbf{x}^{\mathbf{z}^*\mathcal{J}^*\sigma^*}_{j^*-1}\right)_{m_*} = (\mathbf{K}^p)_{m_*}$ with $m^* \in J_*$ or (ii) $\left(\mathbf{x}^{\mathbf{z}^*\mathcal{J}^*\sigma^*}_{j^*-1}\right)_{m^*} = (\mathbf{K})_{m_*}$ with $m^* \notin J_*$. Let us restrict ourselves to the first case, the second is dealt with similarly. Then, as $i^* \ge j_*$, we have

$$\left(\mathbf{x}^{\mathbf{z}^*\mathcal{J}^*\sigma^*}_{i^*}\right)_{m^*} \ge \left(\mathbf{x}^{\mathbf{z}^*\mathcal{J}^*\sigma^*}_{j^*-1}\right)_{m^*} = (\mathbf{K}^p)_{m^*} \quad (5)$$

Also, since $n_* \neq m_*$ and $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma} \in \mathcal{K}^\circ$, we have

$$\left(\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}\right)_{m^*} = \left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}\right)_{m^*} < (\mathbf{K}^p)_{m^*}$$

which is in contradiction to (4) and (5).

This leaves case (i), i.e. $\sigma(1) = \sigma_*(1)$,

$$\mathbf{R}^{J}\left(\mathbf{z} + \mathbf{e}_{\sigma(1)}\right) = \mathbf{R}^{J^*}\left(\mathbf{z}_* + \mathbf{e}_{\sigma_*(1)}\right),$$

$\mathbf{R}^{J}(\mathbf{z}) = \mathbf{R}^{J^*}(\mathbf{z}_*)$, and, in particular, $\mathbf{z} = \mathbf{z}_*$ since $\mathbf{z} \ge \mathbf{0}$ and $\mathbf{z}_* \ge \mathbf{0}$. Further, these results imply that for every $i \in \{2,3,...,n\}$ there is an $i^* \in \{2,3,...,n\}$ such that $\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{x}_{i^*}^{\mathbf{z}^*\mathcal{J}^*\sigma^*}$ and that these equations are equivalent to

$$\sum_{j=2}^{i} \mathbf{R}^{\mathcal{J}}(\mathbf{e}_{\sigma(j)}) = \sum_{j=2}^{i^*} \mathbf{R}^{\mathcal{J}^*}(\mathbf{e}_{\sigma^*(j)}) \quad \text{for } i = 2,3,...,n.$$

Clearly, this is only possible if $J = J_*$ and $\sigma = \sigma_*$.

Express a boundary point as a convex combination

Second, we show that for every $\mathbf{x} \in \partial K$ there is a $\mathbf{z} \in \mathbb{N}_0^n$, $\mathcal{J} \subset \{1,2,\ldots,n\}$, and $\sigma \in S_n$ such that $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma} \in \mathcal{K}^\circ$, $\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} \in \partial\mathcal{K}$ for $i = 1,2,\ldots,n$, and $\mathbf{x} \in \mathrm{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$. We do this by explicitly deriving appropriate $\mathbf{z}$, J, and σ for $\mathbf{x}$.

Define $\mathbf{y} = (|x_1|,|x_2|,...,|x_n|)^T$ and let J be such that $\mathbf{R}_J(\mathbf{x}) = \mathbf{y}$, and then also $\mathbf{R}_J(\mathbf{y}) = \mathbf{x}$. Since $\mathbf{x} \in \partial K$, there is an $n^* \in \{1,2,...,n\}$ such that (i) $x_{n^*} = (\mathbf{K}^p)_{n^*}$ with $n^* \notin J$ or (ii) $x_{n^*} = (\mathbf{K}^m)_{n^*}$ with $n^* \in J$.

Define $\mathbf{z} = (z_1, z_2, \dots, z_n)^T \in \mathbb{N}_0^n$ by

$$z_i := \begin{cases} 0, & \text{if } y_i = 0, \\ y_i - 1 \le z_i < y_i, & \text{if } y_i > 0. \end{cases}$$

for all $i \in \{1,2,...,n\}$. In particular $z_{n^*} := y_{n^*} - 1$ and $\mathbf{K}^m < \mathbf{R}_J(\mathbf{z}) < \mathbf{K}^p$, i.e. $\mathbf{R}_J(\mathbf{z}) \in K^\circ$, by the construction of $\mathbf{z}$ and because $\mathbf{K}^m < \mathbf{0} < \mathbf{K}^p$. Finally, set $\mathbf{w} := \mathbf{y} - \mathbf{z}$. Then $0 \le w_i \le 1$ for all $i = 1,2,...,n$. Let $\sigma \in S_n$ such that $\sigma(1) = n^*$ and

$$1 = w_{\sigma(1)} \ge w_{\sigma(2)} \ge ... w_{\sigma(n)} \ge 0.$$

We define $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{R}^{\mathcal{J}}(\mathbf{z})$. To show that $\mathbf{x} \in \mathrm{co}\left(\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma}, \dots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$ we define

$$\lambda_k = w_{\sigma(k)} - w_{\sigma(k+1)} \ge 0 \text{ for } k = 1,...,n-1 \text{ and } \lambda_n = w_{\sigma(n)} \ge 0.$$

We have $\sum_{i=1}^n \lambda_i = w_{\sigma(1)} = 1$ and $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}$. Indeed, we show that the k-th component of $\sum_{i=1}^n \lambda_i \left(\mathbf{z} + \sum_{j=1}^i \mathbf{e}_{\sigma(j)}\right)$ is y_k, which shows the statement by applying $\mathbf{R}_J$ on both sides.

$$\begin{aligned}\left(\sum_{i=1}^n \lambda_i \left(\mathbf{z} + \sum_{j=1}^i \mathbf{e}_{\sigma(j)}\right)\right)_k &= z_k + \left(\sum_{i=1}^n \lambda_i \sum_{j=1}^i \mathbf{e}_{\sigma(j)}\right)_k \\ &= z_k + \sum_{i=\sigma^{-1}(k)}^n \lambda_i \\ &= z_k + w_{\sigma(\sigma^{-1}(k))} \\ &= z_k + w_k \\ &= y_k\end{aligned}$$

where we have used $\sum_{i=1}^n \lambda_i = 1$. This shows the statement.

Express any point as convex combination

Third, we show that for every $\mathbf{x} \in K$ there is a simplex $S \in T$ such that $\mathbf{x} \in S$. If $\mathbf{x} = \mathbf{0}$ this is obvious. If $\mathbf{x} \neq \mathbf{0}$ there is a $\gamma \ge 1$ such that $\gamma\mathbf{x} \in \partial K$. Above we showed that this implies that $\gamma\mathbf{x}$ can be written as a convex combination,

$$\gamma\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}, \text{ from which } \mathbf{x} = (1-\gamma^{-1})\mathbf{0} + \sum_{i=1}^n (\lambda_i\gamma^{-1})\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}$$

follows.

Remark 8. In Lemma 2.6 we considered simplices in (3) with one vertex in $\mathcal{K}^\circ$ and all other vertices in $\partial\mathcal{K}$, and we specifically assumed that the vertex inside $\mathcal{K}$ is $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}$. This assumption is no loss of generality, since if a simplex $\mathfrak{S} := \operatorname{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right) \in \mathcal{T}^{\text{std}}$ has one vertex in $\mathcal{K}^\circ$ and all other vertices in $\partial\mathcal{K}$, then the vertex inside $\mathcal{K}$ is necessarily $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}$. To see this observe the following: Let $\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{j=1}^{i} \mathbf{e}_{\sigma(j)}\right) \notin \partial\mathcal{K}$ be the vertex of $\mathfrak{S}$ not lying on the boundary. We want to show that $i = 0$. If $i \neq 0$, then $\mathbf{x}_{i-1}^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} - \mathbf{R}^{\mathcal{J}}\left(\mathbf{e}_{\sigma(i)}\right) \in \partial\mathcal{K}$, so there is an $n^* \in \{1,2,\ldots,n\}$ such that (i) $\left(\mathbf{x}_{i-1}^{\mathbf{z}\mathcal{J}\sigma}\right)_{n^*} = (\mathbf{K}^p)_{n^*}$ with $n^* \notin \mathcal{J}$ or (ii) $\left(\mathbf{x}_{i-1}^{\mathbf{z}\mathcal{J}\sigma}\right)_{n^*} = (\mathbf{K}^m)_{n^*}$ with $n^* \in \mathcal{J}$. Let us consider the first case, the second case is dealt with similarly. Since $\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} \notin \partial\mathcal{K}$, we have $\left(\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}\right)_{n^*} < (\mathbf{K}^p)_{n^*}$, i.e.

$$(\mathbf{K}^p)_{n^*} = \left(\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}\right)_{n^*} - 1 < (\mathbf{K}^p)_{n^*} - 1 \quad \text{if } \sigma(i) = n^*$$

or $(\mathbf{K}^p)_{n^*} = \left(\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}\right)_{n^*} < (\mathbf{K}^p)_{n^*}$ if $\sigma(i) \neq n^*$.

In both cases we obtain a contradiction.

Definition 2.7. For an n-simplex $\mathfrak{S} = \operatorname{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ we define its **shape matrix** $X_{\mathfrak{S}} \in \mathbb{R}^{n\times n}$ through

$$X_{\mathfrak{S}} := (\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0)^T$$

Thus, the matrix $X_{\mathfrak{S}}$ is defined by writing the entities of the vector $\mathbf{x}_i - \mathbf{x}_0$ in the i-th row of $X_{\mathfrak{S}}$ for $i = 1,2,\ldots,n$.

For a triangulation $\mathcal{T}$ given as a collection of simplices with ordered vertices we refer to the set $\{X_{\mathfrak{S}} : \mathfrak{S} \in \mathcal{T}\}$ as the shape matrices of the triangulation $\mathcal{T}$.

Remark 9. Definition 2.7 is the reason why we defined a simplex as the convex combination of the vectors in an ordered tuple. The resulting simplex is not dependent on the particular order of the vectors, however, the shape matrix is.

Remark 10. Notice, that because $\mathfrak{S}$ is an n-simplex, the vectors $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ are affinely independent, i.e. the shape matrix $X_{\mathfrak{S}}$ is nonsingular.

Lemma 2.8. *The set of the shape matrices of* $\mathcal{T}^{std}$ *is finite. For any fixed* $K \in \mathbb{N}_0$ *and* $b > 0$ *the set of the shape matrices of* $\mathcal{T}_{K,b}^{std}$ *is finite.*

Proof. Notice that $S_{\mathbf{z}J\sigma}$ and $S_{\mathbf{z}_*J_*\sigma_*}$ have the same shape matrix if $J = J_*$ and $\sigma = \sigma_*$. As there are 2^n different subsets $J \subset \{1,2,...,n\}$ and $n!$ different permutations σ of $\{1,2,...,n\}$ there can be no more than $2^n n!$ different shape matrices for

$\mathcal{T}^{std}$.

The second statement of the lemma now follows immediately, because the simplicial fan at the origin in $\mathcal{T}_{K,b}^{std}$ is finite.

Now we can formulate our linear programming feasibility problem for the system (1). It is followed by several explanatory and clarifying remarks.

Definition 2.9 (The linear programming problem.) Consider the system (1). Let $F > 0$ be a real number and $2 \le N_I < N_O < N_D$ be natural numbers. Define

$I := N_I \cdot F,$ $O := N_O \cdot F,$ and $D := N_D \cdot F$

and the hypercubes

$\mathcal{D} := [-D,D]^n$, $\mathcal{O} := [-O,O]^n$, $\mathcal{I} := [-I,I]^n$, and $\mathcal{F} := [-F,F]^n$.

Let the numbers $2 \le N_I < N_O < N_D$ be chosen such that $\mathbf{x} \in \mathcal{F}$ implies $\mathbf{g}(\mathbf{x}) \in \mathcal{I}$ and $\mathbf{x} \in \mathcal{O}$ implies $\mathbf{g}(\mathbf{x}) \in \mathcal{D}$, i.e.

$$\max_{\|\mathbf{x}\|_\infty \le F} \|\mathbf{g}(\mathbf{x})\|_\infty \le I \quad \text{and} \quad \max_{\|\mathbf{x}\|_\infty \le O} \|\mathbf{g}(\mathbf{x})\|_\infty \le D. \tag{6}$$

Clearly $\mathcal{D} \supset \mathcal{O} \supset \mathcal{I} \supset \mathcal{F}$ and $\mathcal{F}$ contains the origin as an inner point.

Let $K \in \mathbb{N}_0$ and consider the triangulation $\mathcal{T}_{K,F}^{std}$ of $\mathbb{R}^n$ from Definition 2.5. Define

$$\mathcal{T} := \{\mathfrak{S} \in \mathcal{T}_{K,F}^{std} : \mathfrak{S} \cap \mathcal{D}^\circ \neq \emptyset\}. \tag{7}$$

Then, by the definitions of $\mathcal{T}_{K,F}^{std}$ and $\mathcal{D}$, clearly $\bigcup_{\mathfrak{S} \in \mathcal{T}} \mathfrak{S} = \mathcal{D}$ and $\mathcal{T}$ is a triangulation of $\mathcal{D}$ in the sense of Definition 2.3. Before we present the linear programming problem we need a few specifications and definitions.

With $A := D\mathbf{g}(\mathbf{0})$ as the Jacobi matrix of $\mathbf{g}$ at the origin and $Q \in \mathbb{R}^{n \times n}$ an arbitrary positive definite matrix, we solve the discrete time Lyapunov equation

$$A^T PA = P - Q \tag{8}$$

for a positive definite $P \in \mathbb{R}^{n\times n}$, cf. Remark 12. We define

$$V_P(\mathbf{x}) := \|\mathbf{x}\|_P, \tag{9}$$

$$\alpha := \frac{1}{8}\sqrt{\lambda^Q_{\min}/\lambda^P_{\max}}, \tag{10}$$

, (11) $$H_{\max} := \frac{\lambda^P_{\max}}{\sqrt{\lambda^P_{\min}}}\left(1 + \frac{\lambda^P_{\max}}{\lambda^P_{\min}}\right)$$

for every $S_\nu \in T$ define

$$h_\nu := \max_{\mathbf{x},\mathbf{y}\in S_\nu} \|\mathbf{x} - \mathbf{y}\|_2 \tag{12}$$

and B_ν and G_ν letbe constants fulfilling

$$B_\nu \geq n \cdot \max_{\substack{m,r,s=1,2,\ldots,n\\ \mathbf{z}\in\mathfrak{S}_\nu}} \left|\frac{\partial^2 g_m}{\partial x_r \partial x_s}(\mathbf{z})\right| \quad \text{if } S_\nu \subset F \text{ and}$$

$$G_\nu \geq n \cdot \max_{\substack{i,j=1,2,\ldots,n\\ \mathbf{z}\in\mathfrak{S}_\nu}} \left|\frac{\partial g_i}{\partial x_j}(\mathbf{z})\right| \tag{13}$$

if $S_\nu \subset O$. (14)

See Remark 11 for an interpretation of the constants B_ν and G_ν.

We further define

$$h_{I\setminus F} := \max\{h_\nu : S_\nu \subset I \setminus F^\circ\}, \tag{15}$$

$$h_{\partial F,P} := \max\{\|\mathbf{x} - \mathbf{y}\|_P : \mathbf{x} \neq \mathbf{0} \text{ and } \mathbf{y} \neq \mathbf{0} \text{ vertices of an } S_\nu \subset F\}, \tag{16}$$

$$G_F := \max\{G_\nu : S_\nu \subset F\}, \text{and} \tag{17}$$

$$E_F := G_F \cdot \max\{H_{\max} \cdot (h_{I\setminus F})^2/F, 2h_{\partial F,P}\}. \tag{18}$$

$\alpha, H_{\max}, h_\nu, B_\nu, G_\nu, h_{\mathcal{I}\setminus\mathcal{F}}, h_{\partial\mathcal{F},P}, G_{\mathcal{F}}, E_{\mathcal{F}}$

Note that all the constants are strictly positive.

We are now ready to state the linear programming problem. The variables of the linear programming problem are C and $V_{\mathbf{x}}$ for all vertices $\mathbf{x}$ of all of the simplices $S_1, S_2, \ldots, S_N$ in T . The variable C is an upper bound on the gradient of the function $V : D \to \mathbb{R}$ and for every vertex $\mathbf{x}$; the variable $V_{\mathbf{x}}$ is its value at $\mathbf{x}$, i.e. $V(\mathbf{x}) = V_{\mathbf{x}}$, cf. Definition 2.4.

The constraints of the linear programming problem are:

(I) For every $S_\nu = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in T$, $S_\nu \subset I$, we set

$V_{\mathbf{x}_i} = V_P(\mathbf{x}_i)$ for $i = 0, 1, \ldots, n$,

where V_P is the local Lyapunov function from (9).

(II) For every $S_\nu = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in T$ we demand

$$V_{\mathbf{x}_i} \geq V_P(\mathbf{x}_i) \text{ for } i = 0, 1, \ldots, n. \tag{19}$$

(III) For every $S_\nu = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ we define the vectors

$\mathbf{w}_\nu := (V_{\mathbf{x}_1} - V_{\mathbf{x}_0}, V_{\mathbf{x}_2} - V_{\mathbf{x}_0}, \ldots, V_{\mathbf{x}_n} - V_{\mathbf{x}_0})^T$ and $\nabla V_\nu := X_{\mathfrak{S}_\nu}^{-1}\mathbf{w}_\nu$,

where X_{S_ν} is the shape matrix of S_ν, cf. Definition 2.7, and we demand

$$\|\nabla V_\nu\|_1 \leq C. \tag{20}$$

These constraints are linear in the variables of the linear programming problem, cf. Remark 13.

(IV) For every $S_\nu = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in T$, $S_\nu \subset O$, and every $i = 0, 1, \ldots, n$, there is a simplex $S_\mu \in \mathrm{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n) \in T$ such that $\mathbf{g}(\mathbf{x}_i) \in S_\mu$. This means that we can write $\mathbf{g}(\mathbf{x}_i)$ uniquely as a convex combination $\mathbf{g}(\mathbf{x}_i) = \sum_{j=0}^n \mu_j \mathbf{y}_j$ of the vertices of S_μ, cf. Remark 14.

If $S_\nu \subset O \setminus F^\circ$ we demand

$$\sum_{j=0}^{n} \mu_j V_{\mathbf{y}_j} - V_{\mathbf{x}_i} + C G_\nu h_\nu \leq -\alpha \|\mathbf{x}_i\|_Q \quad \text{for } i = 0, 1, \ldots, n. \tag{21}$$

If $S_\nu \subset F$ we demand

$$\sum_{j=0}^{n} \mu_j V_{\mathbf{y}_j} - V_{\mathbf{x}_i} + C B_\nu h_\nu \|\mathbf{x}_i\|_2 + E_{\mathcal{F}} \leq -\alpha \|\mathbf{x}_i\|_Q \quad \text{for } i = 1, \ldots, n. \tag{22}$$

Note, that we do not demand (22) for $i = 0$, cf. Remark 14.

We have several remarks before we prove in Theorem 2.11 that a feasible solution to the linear programming problem in Definition 2.9 parameterizes a CPA Lyapunov function for the system in question. For some of the remarks and for later we need the following results, proved, e.g., in Proposition 4.1 and Lemma 4.2 in [3].

Proposition 2.10. *Let* $\mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k) \subset \mathbb{R}^n$ *be a k-simplex, define* $S := \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k)$, $h := \max_{i,j=0,1,\ldots,k} \|\mathbf{x}_i - \mathbf{x}_j\|_2$, *and consider a convex combination* $\sum_{i=0}^k \lambda_i \mathbf{x}_i \in \mathfrak{S}$. *Let* $U \subset \mathbb{R}^n$ *be an open set with* $S \subset U$.

a) If $g : U \to \mathbb{R}$ *is Lipschitz-continuous with constant L on* U*, i.e.* $|g(\mathbf{x}) - g(\mathbf{y})| \leq L\|\mathbf{x} - \mathbf{y}\|_2$ *for all* $\mathbf{x}, \mathbf{y} \in U$*, then*

$$\left| g\left(\sum_{i=0}^{k} \lambda_i \mathbf{x}_i\right) - \sum_{i=0}^{k} \lambda_i g(\mathbf{x}_i)\right| \le Lh.$$

b) If $g \in C^2(\mathrm{U},\mathrm{R})$ *and* $B_H := \max_{\mathbf{z}\in \mathfrak{S}} \|H(\mathbf{z})\|_2$, *where* $H(\mathbf{z})$ *is the Hessian of* g *at* $\mathbf{z}$,

then

$$\begin{aligned}
&\left| g\left(\sum_{i=0}^{k} \lambda_i \mathbf{x}_i\right) - \sum_{i=0}^{k} \lambda_i g(\mathbf{x}_i)\right| \\
&\qquad \le \frac{1}{2}\sum_{i=0}^{k} \lambda_i B_H \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left(\max_{\mathbf{z}\in\mathfrak{S}} \|\mathbf{z}-\mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2\right) \\
&\qquad \le h B_H \sum_{i=0}^{k} \lambda_i \|\mathbf{x}_i - \mathbf{x}_0\|_2 \\
&\qquad \le B_H h^2,
\end{aligned}$$

Further useful bounds are obtained by noting that

$$B_H \le n \cdot \max_{\substack{r,s=1,2,\dots,n\\ \mathbf{z}\in\mathfrak{S}_\nu}} \left| \frac{\partial^2 g}{\partial x_r \partial x_s}(\mathbf{z})\right|.$$

Remark 11. For every $S_\nu = \mathrm{co}(\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_n) \in \mathrm{T}$, $S_\nu \subset \mathrm{F}$, we have by Proposition 2.10 with the constants B_ν in (13) for every convex combination $\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i$:

$$\left\| \mathbf{g}(\mathbf{x}) - \sum_{i=0}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i)\right\|_\infty \le \frac{1}{2} B_\nu \sum_{i=0}^{n} \lambda_i \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left(h_\nu + \|\mathbf{x}_i - \mathbf{x}_0\|_2\right) \le B_\nu h_\nu^2 .$$

Now let $S_\nu \in \mathrm{O}$. For an interpretation of the constants G_ν in (14), notice that for any $\mathbf{x},\mathbf{y} \in \mathrm{R}^n$ there is an $i \in \{1,2,\dots,n\}$ and a vector $\mathbf{z}_{\mathbf{xy}}$ on the line segment between $\mathbf{x}$ and $\mathbf{y}$ such that

$$\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\|_\infty = |g_i(\mathbf{x}) - g_i(\mathbf{y})| = |\nabla g_i(\mathbf{z}_{\mathbf{xy}}) \cdot (\mathbf{x}-\mathbf{y})| \le \|\nabla g_i(\mathbf{z}_{\mathbf{xy}})\|_1 \|\mathbf{x}-\mathbf{y}\|_\infty.$$

Hence, we have for $S_\nu \in \mathrm{O}$

$$\sup_{\substack{\mathbf{x},\mathbf{y}\in\mathfrak{S}_\nu\\ \mathbf{x}\neq\mathbf{y}}} \frac{\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\|_\infty}{\|\mathbf{x}-\mathbf{y}\|_\infty} \le n \cdot \max_{\substack{i,j=1,2,\dots,n\\ \mathbf{z}\in\mathfrak{S}_\nu}} \left|\frac{\partial g_i}{\partial x_j}(\mathbf{z})\right| \le G_\nu . \qquad (23)$$

Now let $S_\nu \in \mathrm{F}$. In particular, since $\mathbf{g}(\mathbf{0}) = \mathbf{0}$, we have for every $\mathbf{x} \in S_\nu \subset \mathrm{F}$, $\mathbf{x} \neq \mathbf{0}$, that

$$\frac{\|\mathbf{g}(\mathbf{x})\|_\infty}{\|\mathbf{x}\|_\infty} \le G_\nu \le G_{\mathcal{F}} \quad \text{and} \quad \|\mathbf{g}(\mathbf{x})\|_\infty \le G_{\mathcal{F}} \|\mathbf{x}\|_\infty. \qquad (24)$$

Remark 12. The equation (8) is called the discrete Lyapunov equation. For its properties cf. e.g. Lemma 5.7.19 in [38]. It can be solved numerically in an efficient way [6]. See also [29] and [5].

Remark 13. Consider a simplex $S_\nu = \mathrm{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \cdots, \mathbf{x}_n^\nu)$ in the triangulation T . The components of the vector ∇V_ν are linear in the variables $V_{\mathbf{x}_0^\nu}, V_{\mathbf{x}_1^\nu}, \ldots, V_{\mathbf{x}_n^\nu}$ and by introducing the auxiliary variables $C_1^\nu, C_2^\nu, \ldots, C_n^\nu$ it is easily seen that $\|\nabla V_\nu\|_1 \le C$ can be implemented by the constraints

$C_1^\nu + C_2^\nu + \ldots + C_n^\nu \le C$ and $-C_i^\nu \le (\nabla V_\nu)_i \le C_i^\nu$ for $i = 1, 2, \ldots, n$, where $(\nabla V_\nu)_i$ is the i-th component of ∇V_ν. There are several different reasonable ways to force the linear constraints $\|\nabla V_\nu\|_1 \le C$ for $\nu = 1,2,\ldots,N$.

One is to set $C_i^\nu = C/n$ for $i = 1, 2, \ldots, n$ and $\nu = 1, 2, \ldots, N$. In this case, there are no auxiliary variables needed and we will do this in the proof of Theorem 4.2, where we show that we can always compute a CPA Lyapunov function if the equilibrium at the origin is exponentially stable.

The other extreme is to include all the auxiliary variables C_i^ν, $i = 1,2,\ldots,n$ and $\nu = 1,2,\ldots,N$, in the linear programming problem. Here, one might succeed in computing a CPA Lyapunov function with larger simplices than when using fewer or no auxiliary variables.

In between these two extremes one could e.g. neglect the ν dependence of the auxiliary variables C_i^ν and merely introduce the auxiliary variables $C_1,C_2,\ldots,C_n$ and implement the constraints $\|\nabla V_\nu\|_1 \le C$ by $C_1 + C_2 + \ldots + C_n \le C$ and $-C_i \le (\nabla V_\nu)_i \le C_i$ for $i = 1,2,\ldots,n$ and $\nu = 1,2,\ldots,N$.

Remark 14. Consider the constraints (IV) in Definition 2.9. Clearly $\mathbf{g}(\mathbf{x}_i)$ can be in more than one simplex of T . However, the representation $\sum_{j=0}^n \mu_j V_{\mathbf{y}_j}$ in (21) and (22) does not depend on the particular simplex $S_\mu = \mathrm{co}(\mathbf{y}_0,\mathbf{y}_1,\ldots,\mathbf{y}_n)$ such that $\mathbf{g}(\mathbf{x}_i) = \sum_{j=0}^n \mu_j \mathbf{y}_j$ because T is a triangulation. Further, (22) cannot be fulfilled for $i = 0$ because $E_F > 0$.

We now prove that a feasible solution to the linear programming problem in Definition 2.9 parameterizes a CPA Lyapunov function for the system in question.

Theorem 2.11. *If the linear programming problem from Definition 2.9 has a feasible solution, i.e. the variables C and $V_{\mathbf{x}}$ have values such that all the constraints are fulfilled, then the function $V : \mathcal{D} \to \mathbb{R}$, parameterized using the values $V_{\mathbf{x}}$ and the triangulation* T *as in*

Definition 2.4, is a Lyapunov function in the sense of Definition 2.2 for the system (1) used to construct the linear programming problem.

Proof. For every $\mathbf{x} \in \mathrm{D}$ there is a $\mathrm{co}(\mathbf{x}_0,\mathbf{x}_1,...,\mathbf{x}_n) \in \mathrm{T}$ such that $\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i$.

The convexity of the norm $\|\cdot\|_P$ immediately delivers

$$V(\mathbf{x}) = V\left(\sum_{i=0}^{n} \lambda_i \mathbf{x}_i\right) = \sum_{i=0}^{n} \lambda_i V_{\mathbf{x}_i} \geq \sum_{i=0}^{n} \lambda_i \|\mathbf{x}_i\|_P \geq \left\|\sum_{i=0}^{n} \lambda_i \mathbf{x}_i\right\|_P = \|\mathbf{x}\|_P$$

and the definition of V as a piecewise affine function such that $V(\mathbf{0}) = 0$ renders the existence of a constant $b > 0$ such that $V(\mathbf{x}) \leq b\|\mathbf{x}\|_P$ for all $\mathbf{x} \in \mathrm{D}$ obvious. The demanding part of the proof is to show that $V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \leq -\alpha\|\mathbf{x}\|_Q$ for all $\mathbf{x} \in \mathrm{O}$.

To do this we first show the auxiliary result that $|V(\mathbf{z}) - V(\mathbf{y})| \leq C\|\mathbf{z} - \mathbf{y}\|_\infty$ for all $\mathbf{y},\mathbf{z} \in \mathrm{D}$. Define $\mathbf{r}_\mu := \mathbf{y} + \mu(\mathbf{z} - \mathbf{y})$ for all $\mu \in [0,1]$. Since D is convex, the line segment $\{\mathbf{r}_\mu : \mu \in [0,1]\}$ is contained in D and clearly there are numbers

$0 = \mu_0 < \mu_1 < \mu_2 < ... < \mu_K = 1$ and $\nu_1,\nu_2,...,\nu_K$ such that $\mathbf{r}_\mu \in \mathrm{S}_{\nu_i}$ for all

$\mu \in [\mu_{i-1},\mu_i]$, $i = 1,2,...,K$. Now $\mathbf{r}_1 = \mathbf{z}$ and $\mathbf{r}_0 = \mathbf{y}$ and for every $i = 1,2,...,K$ we have $V(\mathbf{x}) = \nabla V_{\nu_i} \cdot (\mathbf{x} - \mathbf{x}_0^{\nu_i}) + V_{\mathbf{x}_0^{\nu_i}}$ for $\mathbf{x} \in \mathfrak{S}_{\nu_i} = \mathrm{co}\left(\mathbf{x}_0^{\nu_i}, \mathbf{x}_1^{\nu_i}, \ldots, \mathbf{x}_n^{\nu_i}\right)$. Thus, by (20),

$$\begin{aligned}
|V(\mathbf{z}) - V(\mathbf{y})| &= \left|\sum_{i=1}^{K} [V(\mathbf{r}_i) - V(\mathbf{r}_{i-1})]\right| \leq \sum_{i=1}^{K} |\nabla V_{\nu_i} \cdot (\mathbf{r}_i - \mathbf{r}_{i-1})| \\
&\leq \sum_{i=1}^{K} \|\nabla V_{\nu_i}\|_1 \|\mathbf{r}_i - \mathbf{r}_{i-1}\|_\infty \leq \sum_{i=1}^{K} C(\mu_i - \mu_{i-1})\|\mathbf{z} - \mathbf{y}\|_\infty \\
&= (\mu_K - \mu_0)C\|\mathbf{z} - \mathbf{y}\|_\infty = C\|\mathbf{z} - \mathbf{y}\|_\infty.
\end{aligned} \tag{25}$$

A direct consequence is that if $\mathbf{y},\mathbf{z} \in \mathrm{S}_\nu \subset \mathrm{O}$, then $\mathbf{g}(\mathbf{y}),\mathbf{g}(\mathbf{z}) \in \mathrm{D}$ and by (23)

$$|V(\mathbf{g}(\mathbf{z})) - V(\mathbf{g}(\mathbf{y}))| \leq C\|\mathbf{g}(\mathbf{z}) - \mathbf{g}(\mathbf{y})\|_\infty \leq CG_\nu\|\mathbf{z} - \mathbf{y}\|_\infty \leq CG_\nu h_\nu. \tag{26}$$

We now show that $V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \leq -\alpha\|\mathbf{x}\|_Q$ for all $\mathbf{x} \in \mathrm{O}$. We first show this for all $\mathbf{x} \in \mathrm{O} \setminus \mathrm{F}^\circ$ and then for all $\mathbf{x} \in \mathrm{F}$.

Case 1: Let $\mathbf{x} \in \mathrm{O} \setminus \mathrm{F}^\circ$ be arbitrary. Then there is an $\mathrm{S}_\nu = \mathrm{co}(\mathbf{x}_0,\mathbf{x}_1,...,\mathbf{x}_n) \subset \mathrm{O} \setminus \mathrm{F}^\circ$ such that $\mathbf{x} \in \mathrm{S}_\nu$, which in turn implies that $\mathbf{x}$ can be written as a convex combination of the vertices of the simplex, $\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i$. But then by (26) and the constraints (21) we have

$$V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) = V(\mathbf{g}(\mathbf{x})) - \sum_{i=0}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i)) + \sum_{i=0}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i)) - \sum_{i=0}^{n} \lambda_i V(\mathbf{x}_i)$$

$$= \sum_{i=0}^{n} \lambda_i \left[V(\mathbf{g}(\mathbf{x})) - V(\mathbf{g}(\mathbf{x}_i)) + V(\mathbf{g}(\mathbf{x}_i)) - V(\mathbf{x}_i)\right]$$

$$\leq \sum_{i=0}^{n} \lambda_i [C G_\nu h_\nu + V(\mathbf{g}(\mathbf{x}_i)) - V(\mathbf{x}_i)]$$

$$\leq -\alpha \sum_{i=0}^{n} \lambda_i \|\mathbf{x}_i\|_Q \leq -\alpha \|\mathbf{x}\|_Q. \qquad (27)$$

Case 2: We now come to the more involved case $\mathbf{x} \in$ F. Let $\mathbf{x} \in$ F be arbitrary. Then there is a simplex $S_\nu = \text{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \subset$ F such that $\mathbf{x} \in S_\nu$ and $\mathbf{x}$ can be written as a convex sum of its vertices, $\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i$. However, now $\mathbf{x}_0 = \mathbf{0}$, which also implies $\mathbf{g}(\mathbf{x}_0) = \mathbf{0}$ and $V(\mathbf{g}(\mathbf{x}_0)) = 0$. Therefore

$$V(\mathbf{x}) = \sum_{i=0}^{n} \lambda_i V(\mathbf{x}_i) = \sum_{i=1}^{n} \lambda_i V(\mathbf{x}_i), \qquad (28)$$

$$\sum_{i=0}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i) = \sum_{i=1}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i), \quad \text{and} \qquad (29)$$

$$\sum_{i=0}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i)) = \sum_{i=1}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i)). \qquad (30)$$

$$V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) = \underbrace{V(\mathbf{g}(\mathbf{x})) - V\left(\sum_{i=1}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i)\right)}_{a)} + \underbrace{V\left(\sum_{i=1}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i)\right) - \sum_{i=1}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i))}_{b)}$$

$$+ \underbrace{\sum_{i=1}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i)) - \sum_{i=1}^{n} \lambda_i V(\mathbf{x}_i)}_{c)}, \qquad (31)$$

We extend $V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x})$ to three differences a), b), and c), namely and then we find upper bounds for a), b), and c) separately. **a)** By (29), (26), and Proposition 2.10 we get

$$\left|V(\mathbf{g}(\mathbf{x}))-V\left(\sum_{i=1}^n \lambda_i \mathbf{g}(\mathbf{x}_i)\right)\right| \leq C\left\|\mathbf{g}(\mathbf{x})-\sum_{i=1}^n \lambda_i \mathbf{g}(\mathbf{x}_i)\right\|_\infty$$
$$= C\left\|\mathbf{g}(\mathbf{x})-\sum_{i=0}^n \lambda_i \mathbf{g}(\mathbf{x}_i)\right\|_\infty$$
$$\leq C\sum_{i=0}^n \lambda_i B_\nu h_\nu \|\mathbf{x}_i-\mathbf{x}_0\|_2$$
$$= CB_\nu h_\nu \sum_{i=1}^n \lambda_i \|\mathbf{x}_i\|_2.$$

(32) **b)** Set $\mathbf{z}_i := \mathbf{g}(\mathbf{x}_i)$ for $i = 0,1,\ldots,n$ and $\mathbf{z} = \sum_{i=0}^n \lambda_i \mathbf{z}_i = \sum_{i=1}^n \lambda_i \mathbf{z}_i$. We show that

$$V(\mathbf{z})-\sum_{i=1}^n \lambda_i V(\mathbf{z}_i) \leq V(\mathbf{z})-V_P(\mathbf{z}) \leq E_{\mathcal{F}}\sum_{i=1}^n \lambda_i. \qquad (33)$$

A norm is a convex function, so V_P, cf. (9), is convex. Using (29) and (30) we get by Jensen's inequality that

$$V_P(\mathbf{z}) = V_P\left(\sum_{i=1}^n \lambda_i \mathbf{z}_i\right) = V_P\left(\sum_{i=0}^n \lambda_i \mathbf{z}_i\right) \leq \sum_{i=0}^n \lambda_i V_P(\mathbf{z}_i) = \sum_{i=1}^n \lambda_i V_P(\mathbf{z}_i). \quad (34)$$

For $i = 1,2,\ldots,n$ we have $\mathbf{z}_i = \mathbf{g}(\mathbf{x}_i) \in \mathfrak{S}_{\nu_i} = \mathrm{co}\left(\mathbf{y}_0^{\nu_i}, \mathbf{y}_1^{\nu_i}, \ldots, \mathbf{y}_n^{\nu_i}\right) \subset \mathcal{I}$ since $\mathbf{x}_i \in \mathcal{F}$.

Thus we can write $\mathbf{z}_i$ as a convex combination of the vertices of $\mathfrak{S}_{\nu_i}$, $\mathbf{z}_i = \sum_{j=0}^n \gamma_j \mathbf{y}_j^{\nu_i}$, and by the definition of V on I (constraint (I)) and Jensen's inequality we get

$$V_P(\mathbf{z}_i) = V_P\left(\sum_{j=0}^n \gamma_j \mathbf{y}_j^{\nu_i}\right) \leq \sum_{j=0}^n \gamma_j \underbrace{V_P\left(\mathbf{y}_j^{\nu_i}\right)}_{=V\left(\mathbf{y}_j^{\nu_i}\right)} = V\left(\sum_{j=0}^n \gamma_j \mathbf{y}_j^{\nu_i}\right) = V(\mathbf{z}_i) \qquad (35)$$

Together, (34) and (35) imply

$$V(\mathbf{z}) - \sum_{i=1}^{n} \lambda_i V(\mathbf{z}_i) \le V(\mathbf{z}) - V_P(\mathbf{z}) + \sum_{i=1}^{n} \lambda_i [V_P(\mathbf{z}_i) - V(\mathbf{z}_i)]$$
$$\le V(\mathbf{z}) - V_P(\mathbf{z}),$$

i.e. the first inequality in (33) holds true.

To prove the second inequality in (33) we first show two auxiliary inequalities, (38) and (40). If $\mathbf{z} \in \mathcal{I} \setminus \mathcal{F}$, then we can use Proposition 2.10 to gain upper bounds on $V(\mathbf{z}) - V_P(\mathbf{z})$. The Hessian matrix of V_P at $\mathbf{z}$ is given by

$$H(\mathbf{z}) = \frac{1}{\|\mathbf{z}\|_P} P - \frac{1}{\|\mathbf{z}\|_P^3} (P\mathbf{z})(P\mathbf{z})^T, \qquad (36)$$

from which, with $H_{\max}$ from (11),

$$\|H(\mathbf{z})\|_2 \le \frac{H_{\max}}{\|\mathbf{z}\|_2} \le \frac{H_{\max}}{F}, \qquad (37)$$

follows. There is an $\mathfrak{S}_\mu = \mathrm{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n) \subset \mathcal{I} \setminus \mathcal{F}^\circ$ such that $\mathbf{z} \in \mathfrak{S}_\mu$ and we can write $\mathbf{z}$ as a convex combination of the vertices of $\mathfrak{S}_\mu$, $\mathbf{z} = \sum_{j=0}^{n} \mu_j \mathbf{y}_j$. Hence, by Proposition 2.10, $\mathbf{z} \in \mathcal{I} \setminus \mathcal{F}$ implies

$$V(\mathbf{z}) - V_P(\mathbf{z}) = V\left(\sum_{j=0}^{n} \mu_j \mathbf{y}_j\right) - V_P(\mathbf{z})$$
$$= \sum_{j=0}^{n} \mu_j V_P(\mathbf{y}_j) - V_P\left(\sum_{j=0}^{n} \mu_j \mathbf{y}_j\right)$$
$$\le \frac{H_{\max}}{F} (h_{\mathcal{I}\setminus\mathcal{F}})^2. \qquad (38)$$

If $\mathbf{z} \in \mathcal{F}$, then there is an $\mathfrak{S}_\mu = \mathrm{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n) \subset \mathcal{F}$ such that $\mathbf{z} \in \mathfrak{S}_\mu$. Define $\mathbf{u}_i := \mathbf{y}_i - \mathbf{y}_1$ for $i = 1, 2, \ldots, n$. We can write $\mathbf{z}$ as a convex combination of the vertices of $\mathfrak{S}_\mu$ and since $\mathbf{y}_0 = \mathbf{0}$ this now implies

$$\mathbf{z} = \sum_{i=0}^{n} \mu_i \mathbf{y}_i = \sum_{i=1}^{n} \mu_i \mathbf{y}_i = \sum_{i=1}^{n} \mu_i (\mathbf{y}_1 + \mathbf{u}_i). \qquad (39)$$

Now

$$V(\mathbf{z}) = \sum_{i=1}^{n} \mu_i \|\mathbf{y}_1 + \mathbf{u}_i\|_P \le \sum_{i=1}^{n} \mu_i \left(\|\mathbf{y}_1\|_P + \|\mathbf{u}_i\|_P\right) \le \sum_{i=1}^{n} \mu_i \left(\|\mathbf{y}_1\|_P + h_{\partial\mathcal{F},P}\right)$$

and

$$V_P(\mathbf{z}) = \left\|\sum_{i=1}^{n} \mu_i (\mathbf{y}_1 + \mathbf{u}_i)\right\|_P \ge \left\|\sum_{i=1}^{n} \mu_i \mathbf{y}_1\right\|_P - \left\|\sum_{i=1}^{n} \mu_i \mathbf{u}_i\right\|_P \ge \sum_{i=1}^{n} \mu_i \left(\|\mathbf{y}_1\|_P - h_{\partial\mathcal{F},P}\right)$$

Hence, $\mathbf{z} \in \mathcal{F}$ implies

$$V(\mathbf{z}) - V_P(\mathbf{z}) \le 2h_{\partial\mathcal{F},P} \sum_{i=1}^{n} \mu_i \le 2h_{\partial\mathcal{F},P}. \qquad (40)$$

We now prove the second inequality in (33), considering two complementary cases: $\sum_{i=1}^{n}\lambda_i > G_{\mathcal{F}}^{-1}$ and $\sum_{i=1}^{n}\lambda_i \le G_{\mathcal{F}}^{-1}$. If $\sum_{i=1}^{n}\lambda_i > G_{\mathcal{F}}^{-1}$, then by (38), $\sum_{i=1}^{n}\mu_i \le$ (40), 1, and the definition of E we have
F

$$V(\mathbf{z}) - V_P(\mathbf{z}) \le \max\left\{\frac{H_{\max}}{F}(h_{\mathcal{I}\setminus\mathcal{F}})^2, 2h_{\partial\mathcal{F},P}\right\} < E_{\mathcal{F}}\sum_{i=1}^{n}\lambda_i \qquad (41)$$

If $\sum_{i=1}^{n}\lambda_i \le G_{\mathcal{F}}^{-1}$, it follows from

$$\|\mathbf{z}\|_\infty = \left\|\sum_{i=1}^{n}\lambda_i\mathbf{g}(\mathbf{x}_i)\right\|_\infty \le \sum_{i=1}^{n}\lambda_i\|\mathbf{g}(\mathbf{x}_i)\|_\infty \le \sum_{i=1}^{n}\lambda_i G_{\mathcal{F}}F \qquad (42)$$

by (24) that $\|\mathbf{z}\|$ $\le F$, i.e. $\mathbf{z} \in$ F. Thus, we can write $\mathbf{z}$ as in formula (39). Note ∞
that the vertices $\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n$ in that formula are not only in the boundary of F = $[-F,F]^n$, a paraxial hypercube, but are also all points at the same side, i.e. there is an $n^* \in \{1,2,\ldots,n\}$ such that $(\mathbf{y}_i)_{n^*} = F$ for all $i = 1,2,\ldots,n$ or $(\mathbf{y}_i)_{n^*} = -F$ for all $i = 1,2,\ldots,n$ (cf. similar argument at the beginning of the proof of Lemma 2.6). Therefore,

$$\|\mathbf{z}\|_\infty = \left\|\sum_{i=1}^{n}\mu_i\mathbf{y}_i\right\|_\infty = \sum_{i=1}^{n}\mu_i\|\mathbf{y}_i\|_\infty = \sum_{i=1}^{n}\mu_i F,$$

which together with (42) implies

$$\sum_{i=1}^{n}\mu_i \le G_{\mathcal{F}}\sum_{i=1}^{n}\lambda_i \, .$$

Hence, by (40) and the definition of E we get F

$$V(\mathbf{z}) - V_P(\mathbf{z}) \le 2h_{\partial\mathcal{F},P}\sum_{i=1}^{n}\mu_i \le 2h_{\partial\mathcal{F},P}G_{\mathcal{F}}\sum_{i=1}^{n}\lambda_i \le E_{\mathcal{F}}\sum_{i=1}^{n}\lambda_i \, .$$

This inequality and (41) prove the second inequality in (33). **c)** The constraints (22) imply

$$\sum_{i=1}^{n}\lambda_i V(\mathbf{g}(\mathbf{x}_i)) - \sum_{i=1}^{n}\lambda_i V(\mathbf{x}_i) = \sum_{i=1}^{n}\lambda_i\left[V(\mathbf{g}(\mathbf{x}_i)) - V(\mathbf{x}_i)\right] \qquad (43)$$

$$\le -\sum_{i=1}^{n}\lambda_i\left[CB_\nu h_\nu\|\mathbf{x}_i\|_2 + E_{\mathcal{F}} + \alpha\|\mathbf{x}_i\|_Q\right] \, .$$

We now finish the proof by applying the results from a), b), and c), i.e. (32), (33), and (43), to (31) and obtain

$$V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \leq \underbrace{CB_\nu h_\nu \sum_{i=1}^{n} \lambda_i \|\mathbf{x}_i\|_2}_{a)} + \underbrace{E_{\mathcal{F}} \sum_{i=1}^{n} \lambda_i}_{b)}$$

$$\underbrace{- \sum_{i=1}^{n} \lambda_i \left[CB_\nu h_\nu \|\mathbf{x}_i\|_2 + E_{\mathcal{F}} + \alpha \|\mathbf{x}_i\|_Q \right]}_{c)}$$

$$\leq -\alpha \sum_{i=1}^{n} \lambda_i \|\mathbf{x}_i\|_Q \leq -\alpha \left\| \sum_{i=1}^{n} \lambda_i \mathbf{x}_i \right\|_Q = -\alpha \|\mathbf{x}\|_Q \tag{44}$$

Remark 15. One might be tempted to assume that the CPA approximation of a convex function is also convex. As this would imply that the term b) in (31) was negative, the factor E in the constraints (22) would not be necessary and the proof F

of Theorem 2.11 would be much shorter. However, in general this is not true as shown by the following counterexample:

Consider the convex function $P(x,y)$ 7→ $(x\ y)$(0 1)(y)

3 0 x

and triangles with vertices (0,2),(−1,1),(1,1) (0,0),(−1,1),(1,1). the ~ of P on these triangles we have $\widetilde{P}(0,2) = P(0,$ $P(0,1) = 0.5 \cdot P(-1,1) + 0.5 \cdot P(1,1) = 4$. Thus the and $5 \cdot P(0,0) < P(0.5 \cdot 0 + 0.5 \cdot 0, 0.5 \cdot 2 + 0.5 \cdot 0) = P(0,1) = 4$ For

CPA approximation P2) = 4,

$\widetilde{}(0,0) = P$

$2 = 0.5 \cdot$ ~ ~

$P$and P is not convex. (0,0) = 0 but *Remark 16*e . It remains an interesting question

if the convexity of the CPA approx-e e P(0,2) + 0. and the triangulation in a useful way for

our application.e imation P of a convex function P can be characterized in terms of the

function P

Remark 17. A practical note for the implementation of the linear programming problem: Theorem 2.11 still holds true if (10), (11), (12), (15), (16), and (17) are replaced by

,

$$0 < \alpha < \frac{1}{8}\sqrt{\lambda_{\min}^{Q}/\lambda_{\max}^{P}},$$ $\nu \geq \max_{\mathbf{x},\mathbf{y}\in\mathfrak{S}_\nu} \|\mathbf{x}-\mathbf{y}\|_2$,

$$H_{\max} \geq \frac{\lambda_{\max}^{P}}{\sqrt{\lambda_{\min}^{P}}}\left(1+\frac{\lambda_{\max}^{P}}{\lambda_{\min}^{P}}\right)$$ $h_{I\setminus F} \geq \max\{h_\nu : S_\nu \subset I \setminus F^\circ\}$, $h_{\partial F,P} \geq \max\{\|\mathbf{x}-\mathbf{y}\|_P$:

h $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$ vertices of an $S \subset F\}$, and $G_F \geq$

$\max\{G_\nu : S_\nu \subset F\}$.

3. The Algorithm

In the next definition we present an algorithm that generates linear programming problems as in Definition 2.9 for the system (1). It starts with a fixed triangulation of a hypercube D ∈ N and refines the triangulation whenever the linear programming problem does not possess a feasible solution. The refinement is such that eventually a linear programming problem is generated, which possesses a feasible solution, whenever the origin is an exponentially stable equilibrium of the system and D is in its basin of attraction. This is proved in Theorem 4.2 in the next section, the main contribution of this paper.

Definition 3.1 (The algorithm.) The main idea of the algorithm is to define a sequence of finer and finer grids, indexed by K. They become finer both near the origin, so a finer and smaller fan, as well as outside. Hence, O and D will not depend

onFor the algorithm we first initialize a few parameters. Let$K$, whereas F_K and I_K do depend on$_n$ K_n. $Q \in \mathbb{R}^{n\times n}$ be an arbitrary, positive definite matrix and let $P \in \mathbb{R}^{\times}$ be the unique solution to the discrete Lyapunov equation (8). We fix a real number $F_0 > 0$ and positive integers $N_{I,0}$,$N_{O,0}$, and $N_{D,0}$. Define

$I_0 := N_{I,0}F_0$, $O_0 := N_{O,0}F_0$, $D_0 := N_{D,0}F_0$,

$I_0 := [-I_0,I_0]^n$, $O := [-O_0,O_0]^n$, $D := [-D_0,D_0]^n$, $F_0 := [-F_0,F_0]^n$. The number $N_{I,0}$ must be chosen such that $N_{O,0} > N_{I,0} \geq 2$ and

$$N_{I,0} \geq n \cdot \max_{\substack{i,j=1,2,\ldots,n\\ \|\mathbf{z}\|_\infty \leq F_0}} \left|\frac{\partial g_i}{\partial x_j}(\mathbf{z})\right|.$$

This last inequality implies

$$I_0 = F_0 N_{I,0} \geq \max_{\|\mathbf{x}\|_\infty \leq F_0} \|\mathbf{g}(\mathbf{x})\|_\infty,$$

cf. Remark 11.

The numbers $N_{O,0}$ and $N_{D,0}$ must be chosen such that $N_{D,0} > N_{O,0}$ and $\mathbf{g}(O) \subset D$, i.e.,

$$\max_{\|\mathbf{x}\|_\infty \le O_0} \|\mathbf{g}(\mathbf{x})\|_\infty \le D_0 . \qquad (45)$$

For all $K \in \mathbb{N}_0$ we define

$F_K := 2^{-K} F_0$, $N_{I,K} := N_{I,0}$, $I_K := N_{I,K} F_K$,

$N_{O,K} := 2^K N_{O,0}$, $O_K := N_{O,K} F_K = N_{O,0} F_0$, $N_{D,K} := 2^K N_{D,0}$,

$D_K := N_{D,K} F_K = N_{D,0} F_0$, $\mathcal{F}_K := [-F_K, F_K]^n$, $\mathcal{I}_K := [-I_K, I_K]^n$.

We fix constants B and G such that

$$B \ge n \cdot \max_{\substack{m,r,s=1,2,\dots,n \\ \mathbf{z}\in\mathcal{D}}} \left| \frac{\partial^2 g_m}{\partial x_r \partial x_s}(\mathbf{z}) \right|$$

and

$$G \ge n \cdot \max_{\substack{i,j=1,2,\dots,n \\ \mathbf{z}\in\mathcal{D}}} \left| \frac{\partial g_i}{\partial x_j}(\mathbf{z}) \right| .$$

Now, for any $K \in \mathbb{N}_0$ we can construct a linear programming problem as in Definition 2.9 with $F := F_K$, $N_I := N_{I,K}$, $N_O := N_{O,K}$, and $N_D := N_{D,K}$. Then the constants I, O, and D in Definition 2.9 are given by $I := I_K$, $O := O_K = O_0$, and $D := D_K = D_0$. Note especially that $F := F_K$ and $I := I_K$ change with K but O and D do not. Thus, (45) holds true with O_0 replaced by O_K and D_0 replaced by D_K.

Further, for all $K \in \mathbb{N}_0$ we have $\mathbf{g}(\mathcal{F}_K) \subset \mathcal{I}_K$ because $F_K \le F_0$ and therefore

$$\max_{\|\mathbf{x}\|_\infty \le F_K} \|\mathbf{g}(\mathbf{x})\|_\infty \le F_K \cdot n \cdot \max_{\substack{i,j=1,2,\dots,n \\ \|\mathbf{z}\|_\infty \le F_K}} \left| \frac{\partial g_i}{\partial x_j}(\mathbf{z}) \right| \le F_K N_{I,0} = I_K .$$

Hence, the matrices Q and P and the parameters $F := F_K$, $N_I := N_{I,K}$, $N_O := N_{O,K}$, and $N_D := N_{D,K}$ are suitable to initialize the linear programming problem in Definition 2.9. Denote by L_K such a linear programming problem initialized with these parameters, the triangulation $\mathcal{T}_K := \mathcal{T}^{std}_{K,F_K}$, and $B_\nu := B$ and $G_\nu := G$ for all simplices $\mathfrak{S}_\nu$ in the triangulation of $\mathcal{D}$ as defined in (7). The algorithm is as follows:

1. Set $K = 0$.
2. Construct the linear programming problem L_K as described above.
3. If the linear programming problem L_K has a feasible solution, then use it to parameterize a CPA Lyapunov function $V : \mathcal{D} \to \mathbb{R}$ for the system (1) as in Theorem 2.11. If the linear programming problem L_K does not have a feasible solution, then increase K by one, i.e. $K \leftarrow K + 1$, and repeat step 2.

Remark 18. If better estimates for the B_ν's and G_ν's than the uniform bounds B and G in the algorithm are available, then these can and should be used.

Remark 19. Note that the scaling factor ρ from item (3) in Definition 2.3 for the simplicial complex $\mathcal{T}_K = \mathcal{T}_{K,F^{std}_K}$ is $\rho = 2^{-K}F_K = 2^{-2K}F_0$.

The number of simplices in the simplicial fan at the origin grows exponentially. Indeed, it is not difficult to see that the simplicial fan of $\mathcal{T}_{K+1}$ contains 2^{n-1}-times the number of simplices in the simplicial fan of $\mathcal{T}_K$.

4. **Main result**

First, we state a fundamental lemma, the results of which are used in the proof of Theorem 4.2, which is the main contribution of this paper. It ensures the existence of a certain Lyapunov function for the systems (1) if the origin is an exponentially stable equilibrium. It states results similar to Theorem 3.3 in [14] for continuous, planar systems, adapted to n-dimensional discrete systems.

Lemma 4.1. *Consider the system (1) and assume that the origin is an exponentially stable equilibrium of the system with basin of attraction* $\mathcal{A}$. *Let* $Q \in \mathbb{R}^{n\times n}$ *be an arbitrary positive definite matrix,* $A := D\mathbf{g}(\mathbf{0})$ *be the Jacobi matrix of* $\mathbf{g}$ *at the origin, and* $P \in \mathbb{R}^{n\times n}$ *be the unique (positive definite) solution to the discrete Lyapunov equation* $A^T PA - P = -Q$. *Let* $\mathcal{D} \in \mathcal{N}$ *be a subset of* $\mathcal{A}$. *Then there exists a function* $W : \mathcal{A} \to \mathbb{R}$ *that satisfies the following conditions*:

a) $\mathcal{A}$ *is an open set and* $W \in C^2(\mathcal{A} \setminus \{\mathbf{0}\}, \mathbb{R})$.

b) There is a constant $C_* < +\infty$ *such that*

$$\sup_{\mathbf{x}\in\mathcal{D}\setminus\{\mathbf{0}\}} \|\nabla W(\mathbf{x})\|_2 \le C_*. \tag{46}$$

c) Set $\varepsilon_* := \min_{\mathbf{x}\in\partial\mathcal{D}} \|\mathbf{x}\|_2$. *For all* $0 < \varepsilon < \varepsilon_*$ *define*

$$A_\varepsilon := \max_{i,j=1,2,\ldots,n} \left\{ \left| \frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{x}) \right| : \mathbf{x} \in \mathcal{D} \setminus \mathcal{B}_\varepsilon \right\}. \tag{47}$$

Then there is a constant $A < +\infty$ *such that*

$$A_\varepsilon \le \frac{A}{\varepsilon} \tag{48}$$

for all $0 < \varepsilon < \varepsilon_*$.

d)

$$W(\mathbf{x}) \ge \|\mathbf{x}\|_P \quad \text{and} \quad W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) \le -2\alpha\|\mathbf{x}\|_Q \tag{49}$$

for all $\mathbf{x} \in \mathcal{D}$. *Here* $\alpha := 1/8 \cdot \sqrt{\lambda^Q_{\min}/\lambda^P_{\max}}$, *i.e. the* α *from (10).*

e) There is a constant $\delta > 0$ *such that*

$$W(\mathbf{x}) = \|\mathbf{x}\|_P \quad \text{for all } \mathbf{x} \in \mathcal{B}_\delta. \tag{50}$$

Proof. The idea of how to construct the function W is as follows: Locally, at the origin, W is given by the formula (50) and away from the origin by the formula $W(\mathbf{x}) := \beta \sum_{k=0}^{+\infty} \|\mathbf{g}^{\circ k}(\mathbf{x})\|_Q,\ \beta >$ 0 a constant. In between, W is a smooth interpolation of these two. First we work this construction out and then we show that the constructed function fulfills the claimed properties a), b), c), d), and e).

For completeness we show that A is open: Since the equilibrium at the origin is exponentially stable, there is an $\epsilon > 0$ such that $B_\epsilon \subset A$. Take an arbitrary $\mathbf{x} \in A$. There is a $k \in \mathbb{N}$ such that $\mathbf{g}^{\circ k}(\mathbf{x}) \in B_{\epsilon/2}$. By the continuity of $\mathbf{g}^{\circ k}$ there is a $\delta > 0$ such that for all $\mathbf{y} \in \mathbf{x} + B_\delta$ we have $\mathbf{g}^{\circ k}(\mathbf{y}) \in \mathbf{g}^{\circ k}(\mathbf{x}) + B_{\epsilon/2} \subset B_\epsilon \subset A$, i.e. $\mathbf{y} \in A$. This would hold equally true if the origin was merely asymptotically stable.

Definition of W: Since P is a solution to the discrete Lyapunov equation (8), it follows immediately that $\widetilde{V}_P(\mathbf{x}) = \|\mathbf{x}\|_P^2$ is a Lyapunov function for the linear system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, satisfying $\widetilde{V}_P(A\mathbf{x}) - \widetilde{V}_P(\mathbf{x}) = -\|\mathbf{x}\|_Q^2$.

Since $\mathbf{g}$ is differentiable at the origin, the function $\psi(\mathbf{x}) := (\mathbf{g}(\mathbf{x}) - A\mathbf{x})/\|\mathbf{x}\|_2$ fulfills $\lim_{\mathbf{x}\to\mathbf{0}}\psi(\mathbf{x}) = \mathbf{0}$. Simple calculations give, with $\psi^*(\mathbf{x}) := \mathbf{g}(\mathbf{x}) - A\mathbf{x} = \|\mathbf{x}\|_2\psi(\mathbf{x})$, that

$$
\begin{aligned}
\widetilde{V}_P(\mathbf{g}(\mathbf{x})) - \widetilde{V}_P(\mathbf{x}) &= [\mathbf{g}(\mathbf{x})]^T P[\mathbf{g}(\mathbf{x})] - \mathbf{x}^T P\mathbf{x} \\
&= [\psi^*(\mathbf{x}) + A\mathbf{x}]^T P[\psi^*(\mathbf{x}) + A\mathbf{x}] - \mathbf{x}^T P\mathbf{x} \\
&= \psi^{*T}(\mathbf{x})P\psi^*(\mathbf{x}) + \psi^{*T}(\mathbf{x})PA\mathbf{x} + \mathbf{x}^T A^T P\psi^*(\mathbf{x}) + \underbrace{\mathbf{x}^T A^T PA\mathbf{x} - \mathbf{x}^T P\mathbf{x}}_{=-\|\mathbf{x}\|_Q^2} \\
&\leq -\|\mathbf{x}\|_Q^2 + \|\psi^*(\mathbf{x})\|_2\|P\|_2\left(\|\psi^*(\mathbf{x})\|_2 + 2\|A\|_2\|\mathbf{x}\|_2\right) \\
&= -\|\mathbf{x}\|_Q^2 + \|\mathbf{x}\|_2^2 \cdot \|\psi(\mathbf{x})\|_2\|P\|_2\left(\|\psi(\mathbf{x})\|_2 + 2\|A\|_2\right)
\end{aligned}
\tag{51}
$$

and it follows that there is a $\delta_* > 0$ such that $\widetilde{V}_P(\mathbf{g}(\mathbf{x})) - \widetilde{V}_P(\mathbf{x}) \leq -\frac{1}{2}\|\mathbf{x}\|_Q^2$ for all $\mathbf{x} \in B_{\delta_*}$. Hence, with $V_P(\mathbf{x}) = \sqrt{\widetilde{V}_P(\mathbf{x})} = \|\mathbf{x}\|_P$ we have, because $\widetilde{V}_P(\mathbf{g}(\mathbf{x})) < \widetilde{V}_P(\mathbf{x})$ and $\|\mathbf{x}\|_Q/\|\mathbf{x}\|_P \geq \sqrt{\lambda_{\min}^Q/\lambda_{\max}^P} = 8\alpha$ for all $\mathbf{x} \in B_{\delta_*} \setminus \{\mathbf{0}\}$, that

$$V_P(\mathbf{g}(\mathbf{x})) - V_P(\mathbf{x}) = \frac{V_P(\ (\ )) - V_P(\ )}{\sqrt{V}} \leq \frac{-\|\ \|_Q^2/2}{V_P(\mathbf{x})} = \frac{-\|\mathbf{x}\|_Q^2}{4\|\mathbf{x}\|_P} \leq -2\alpha\|\mathbf{x}\|_Q$$ for all $\mathbf{x} \in$

$B_{\delta^*} \setminus \{\mathbf{0}\}$. Thus $\mathbf{g}\ \mathbf{x}$ $\mathbf{x}$ $\mathbf{x}$

$$V_P(\ (\ )) - V_P(\ ) \leq -2\alpha\|\mathbf{x}\|_Q \quad \widetilde{V}_P(\mathbf{g}(\mathbf{x})) + \sqrt{\widetilde{V}_P(\mathbf{x})} \quad 2\sqrt{\widetilde{}} \tag{52}$$

for all $\mathbf{x} \in B_{\delta^*}$. $\mathbf{g}\ \mathbf{x}$ $\mathbf{x}$

Consider the function $\widetilde{W} : \mathcal{A} \to \mathbb{R}$,

$$\widetilde{W}(\mathbf{x}) := \sum_{k=0}^{+\infty} \|\mathbf{g}^{\circ k}(\mathbf{x})\|_Q^2. \tag{53}$$

It follows from the exponential stability of the equilibrium that the series on the right-hand side is convergent and in the proof of Theorem 2.8 in [11] it is shown that $\mathbf{g}$) implies $\widetilde{W} \in C^2(\mathcal{A}, \mathbb{R})$. By the definition of $\widetilde{} \in C^2(\mathbb{R}^n, \mathbb{R}^n$ W clearly

$$\widetilde{W}(\mathbf{x}) = \sum_{k=0}^{+\infty} \|\mathbf{g}^{\circ k}(\mathbf{x})\|_Q^2 = \|\mathbf{x}\|_Q^2 + \sum_{k=1}^{+\infty} \|\mathbf{g}^{\circ k}(\mathbf{x})\|_Q^2 \geq \|\mathbf{x}\|_Q^2 \tag{54}$$

and

$$\widetilde{W}(\mathbf{g}(\mathbf{x})) - \widetilde{W}(\mathbf{x}) = \sum_{k=0}^{+\infty}\left(\|\mathbf{g}^{\circ (k+1)}(\mathbf{x})\|_Q^2 - \|\mathbf{g}^{\circ k}(\mathbf{x})\|_Q^2\right) = -\|\mathbf{x}\|_Q^2 \tag{55}$$

for all $\mathbf{x} \in \mathcal{A}$. Now choose an $r > 0$ such that $\{\mathbf{x} \in \mathbb{R}^n : V_P(\mathbf{x}) \leq r\} \subset B_{\delta^*}$ and define the sets $E_1 := \{\mathbf{x} \in \mathbb{R}^n : V_P(\mathbf{x}) < r/2\}$ and

$E_2 := \{\mathbf{x} \in \mathbb{R}^n : V_P(\mathbf{x}) > r\} \cap \mathcal{A}$.

See Figure 2 for a schematic picture of the sets E_1, $D \setminus (E_1 \cup E_2)$, and $E_2 \cap D$ that we will use in the rest of the proof.

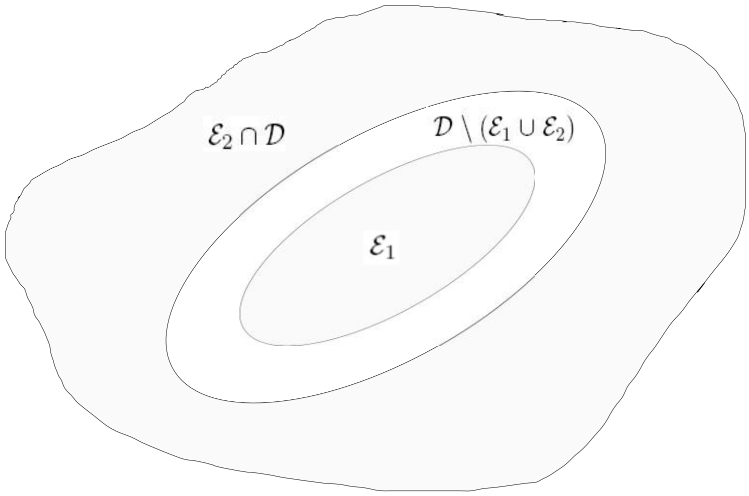


Figure 2. Schematic figure of the sets E1, D \ (E1 ∪ E2), and E2 ∩ D.

$\tilde{} \in C^\infty(\mathbb{R}, [0,$ Let ρ 1]) be a non-decreasing $\widetilde{\rho}(x) = 0$ if $x < r/2$

$\tilde{} \; x) = 1$ if $x > r$ function, such that and $\rho($. Such a $\rho \in C^2(\mathcal{A} \setminus \{\mathbf{0}\}, \mathbb{R}$

function can be constructed by standard methods of $0 \le \rho(\mathbf{x}) \le 1$ for all $\mathbf{x} \in \mathbb{R}^n$, $\rho(\mathbf{x}) = 0$ for all e$\mathbf{x} \in$ E1, and $\rho(\mathbf{x}) = 1$ for all $\mathbf{x} \in$ E2. partitions of unity, cf. e.g. [39]. Then $\rho(\mathbf{x}) := \rho(V_P(\mathbf{x}))$ fulfills),

Define

β := $\overline{P(\mathbf{x})}$ $\overline{2\alpha}$ $\overline{4\alpha \; \mathbf{x} \; Q + r}$ and $\widetilde{W}\;(\mathbf{x}) :=$ $\tilde{}$

$\tilde{}$ implies for all $\mathbf{x} \in \mathcal{D} \setminus \mathcal{E}_1$ that

$$\widetilde{W}_\beta(\mathbf{x}) \ge V_P(\mathbf{x}) = \|\mathbf{x}\|_P,$$

$$\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x}) = -\beta\|\mathbf{x}\|_Q^2 \le -2\alpha\|\mathbf{x}\|_Q,$$

$\frac{r}{2}$ α r $\|_Q$

$$\in\mathcal{D}\setminus\mathcal{E}_1 \max\left\{ \begin{array}{l} V \\ W(\mathbf{x}) \end{array}, \; \|\mathbf{x}\|_Q, \quad 2W(\mathbf{x}) \right\} \quad \beta \quad \beta$$

max‖ ‖

$W(\mathbf{x})$.

Note that this definition ofe β and W_β e x

(56)

and (57)

$- W_\beta(\mathbf{x}) \le 2r - 4 \,\|^{\mathbf{x}}\|2^{Q\,+}$ $-$ $\overline{}$ $= -2\alpha\|\mathbf{x}$. (58)

We define for all $\mathbf{x} \in$ Ae the function W through

$W(\mathbf{x}) := \rho(\mathbf{x})W \quad (\mathbf{x}) \quad + \quad \widetilde{}_{\beta} \quad - (1 \quad \rho(\mathbf{x}))V_P(\mathbf{x}).$ (59)

We will now check that the $W(\mathbf{x}$ function) satisfies the properties a)–e). **a)** Because ρ, V_P, $(\mathcal{A} \setminus \{\mathbf{0}\}, \mathbb{R}$ and We_β are in C^2) then so is W.

b) For every $\mathbf{x} \neq \mathbf{0}$ we have $\nabla V_P(\mathbf{x}) = P\mathbf{x}/\|\mathbf{x}\|_P$ so for every $\mathbf{x} \neq \mathbf{0}$

$$\|\nabla V_P(\mathbf{x})\|_2 = \frac{\|P\mathbf{x}\|_2}{\|\mathbf{x}\|_P} \leq \frac{\lambda^P_{\max}}{\sqrt{\lambda^P_{\min}}} < +\infty.$$

Because ∇W is continuous on the compact set D \ E1 and W and V_P coincide on E1 sup

$\|\nabla W(\mathbf{x})\|_2 = \max\{ \max_1 \|\nabla W(\mathbf{x})\|_2, \sup_1 \|\nabla V_P(\mathbf{x})\|_2\} < +\infty$ $\mathbf{x} \in D \setminus \{\mathbf{0}\}$ $\mathbf{x} \in D \setminus E$ $\mathbf{x} \in E \setminus \{\mathbf{0}\}$ and there is a

constant C_* such that (46) holds true.

c) Denote by $p_{\max}$ the maximum absolute value of the entities of P, i.e. $p_{\max} := \max_{i,j=1,2,\ldots,n} |p_{ij}|$. Define

$$A := \max\left\{\varepsilon^* \cdot \max_{\substack{i,j=1,2,\ldots,n \\ \mathbf{x}\in\mathcal{D}\setminus\mathcal{E}_1}} \left|\frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{x})\right|, \ \frac{1}{\sqrt{\lambda^P_{\min}}}\left(p_{\max} + \frac{(\lambda^P_{\max})^2}{\lambda^P_{\min}}\right)\right\}.$$

For an arbitrary ε, $0 < \varepsilon < \varepsilon^*$, let $\mathbf{y} \in \mathcal{D} \setminus \mathcal{B}_\varepsilon$ and $i, j \in \{1, 2, \ldots, n\}$ be such that

$$A_\varepsilon = \left|\frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{y})\right|.$$

To show (48), we distinguish between the two cases $\mathbf{y} \in$ D \ E1 and $\mathbf{y} \in$ E1. In the first case, (48) clearly holds true because $\varepsilon^*/\varepsilon > 1.$

Now assume that $\mathbf{y} \in$ E1. In this case $W(\mathbf{x})$ coincides with $V_P(\mathbf{x}) = \|\mathbf{x}\|_P$ in a neighbourhood of $\mathbf{y}$ and we have the formula (36) for its Hessian matrix. By definition, A_ε is the maximum of the absolute values of the entities of the Hessian $H_W(\mathbf{x})$ for $\mathbf{x} \in$ D \ Bε and $\|\mathbf{y}\|_2 \geq \varepsilon$ because we have

$$A_\varepsilon = \left|\frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{y})\right| \leq \frac{1}{\sqrt{\lambda^P_{\min}}}\left(\frac{p_{\max}}{\|\mathbf{y}\|_2} + \frac{(\lambda^P_{\max})^2\|\mathbf{y}\|_2^2}{\lambda^P_{\min}\|\mathbf{y}\|_2^3}\right) \leq \frac{A}{\varepsilon}. \tag{60}$$

Hence, estimate (48) holds true for all $0 < \varepsilon < \varepsilon^*$.

d) For all $\mathbf{x} \in$ E1 we have $W(\mathbf{x}) = V_P(\mathbf{x}) = \|\mathbf{x}\|_P$. For all $\mathbf{x} \in$ D\E1 we have by (59) that W is point-wise the convex combination of $\widetilde{W}_\beta$ and V_P. Hence, by (56) we have

$$W(\mathbf{x}) \geq \min\{\widetilde{W}_\beta(\mathbf{x}), V_P(\mathbf{x})\} \geq \|\mathbf{x}\|_P$$

for all $\mathbf{x} \in D \setminus E_1$

and the first estimate in (49) holds true.

To prove the second estimate in (49) we consider three complementary cases, $\mathbf{x} \in E_1$, $\mathbf{x} \in D \setminus (E_1 \cup E_2)$, and $\mathbf{x} \in E_2 \cap D$, cf. Figure 2. The identity

$$W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x})$$

$$= \rho(\mathbf{g}(\mathbf{x}))\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) + (1 - \rho(\mathbf{g}(\mathbf{x})))V_P(\mathbf{g}(\mathbf{x})) - \rho(\mathbf{x})\widetilde{W}_\beta(\mathbf{x}) - (1 - \rho(\mathbf{x}))V_P(\mathbf{x}) \quad (61)$$

$$= \rho(\mathbf{g}(\mathbf{x}))\left[\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x})\right] + (1 - \rho(\mathbf{g}(\mathbf{x})))[V_P(\mathbf{g}(\mathbf{x})) - V_P(\mathbf{x})]$$

$$+ [\rho(\mathbf{g}(\mathbf{x})) - \rho(\mathbf{x})]\left[\widetilde{W}_\beta(\mathbf{x}) - V_P(\mathbf{x})\right]$$

is useful for some of these cases. Further note that

$$\|\mathbf{g}(\mathbf{x})\|_P = V_P(\mathbf{g}(\mathbf{x})) \leq V_P(\mathbf{x}) = \|\mathbf{x}\|_P \quad (62)$$

for all $\mathbf{x} \in D \setminus E_2$

because V_P is a Lyapunov function for the system (1) on $B_{\delta^*} \supset D \setminus E_2$. This implies, because ρ is monotonically increasing,

$$\rho(\mathbf{x}) = \widetilde{\rho}(V_P(\mathbf{x})) \geq \widetilde{\rho}(V_P(\mathbf{g}(\mathbf{x}))) = \rho(\mathbf{g}(\mathbf{x})) \quad \text{for all } \mathbf{x} \in D \setminus E_2 \quad (63)$$

as well as

$$\mathbf{x} \in E_1 \Rightarrow \mathbf{g}(\mathbf{x}) \in E_1 \quad \text{and} \quad \mathbf{x} \in D \setminus E_1 \Rightarrow \mathbf{g}(\mathbf{x}) \in D \setminus E_1. \quad (64)$$

Case 1: Assume $\mathbf{x} \in E_1$, then by (64) and the definition of ρ we have $\rho(\mathbf{x}) = \rho(\mathbf{g}(\mathbf{x})) = 0$ and by (61) and (52) we get

$$W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) = V_P(\mathbf{g}(\mathbf{x})) - V_P(\mathbf{x}) \leq -2\alpha\|\mathbf{x}\|_Q. \quad (65)$$

Case 2 Assume $\mathbf{x} \in D \setminus (E_1 \cup E_2)$. Then by (63) $\rho(\mathbf{g}(\mathbf{x})) - \rho(\mathbf{x}) \leq 0$ and by (56) $\widetilde{W}_\beta(\mathbf{x}) - V_P(\mathbf{x}) \geq 0$ so (62), (57), and (52) deliver

$$W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x})$$

$\leq \rho(\mathbf{g}(\mathbf{x}))\big[W e_\beta(\mathbf{g}(\mathbf{x}))\ \bar{}_\beta W e_\beta(\mathbf{x})\big]_P + (1 - \rho(\mathbf{g}(_P\mathbf{x})))[\}V_P(\mathbf{g}(\mathbf{x}))\ \bar{}_Q V_P(\mathbf{x})] \leq$

$\max\{W e_\beta(\mathbf{g}(\mathbf{x}))_2 - W e(\mathbf{x}), V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \leq -2\alpha\|\mathbf{x}\|$.

Case 3 Assume that $\mathbf{x} \in E \cap D$ until the end of this part of the proof. Here, we consider the three cases $\mathbf{g}(\mathbf{x}) \in E_2 \cap D$, $\mathbf{g}(\mathbf{x}) \in D\backslash(E_1 \cup E_2)$, and $\mathbf{g}(\mathbf{x}) \in E_1$ separately. If $\mathbf{g}(\mathbf{x}) \in E_2 \cap D$, then $\rho(\mathbf{x}) = \rho(\mathbf{g}(\mathbf{x})) = 1$ and (61) and (57) imply

$$W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) = \widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x}) \leq -2\alpha\|\mathbf{x}\|_Q.$$

If $\mathbf{g}(\mathbf{x}) \in \mathcal{D} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$ we have $\rho(\mathbf{g}(\mathbf{x})) - \rho(\mathbf{x}) = \rho(\mathbf{g}(\mathbf{x})) - 1 \leq 0$ and by (56) $\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) \geq V_P(\mathbf{g}(\mathbf{x}$)). We can use this to simplify (62) and then use (57) to estimate from above,

$W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x})$

$= \rho(\mathbf{g}(\mathbf{x}))\big[W^{e}{}_{\beta\beta}(\mathbf{g}(\mathbf{x})) - W^{e}{}_{\beta\beta}(\mathbf{x})\big] + (1 - \rho(\mathbf{g}(\mathbf{x})))\big[[V e^{P}{}_\beta(\mathbf{g}(\mathbf{x})) - W^{e} e^{\beta}{}_\beta(\mathbf{x})\big]]$

$\leq \rho(\mathbf{g}(\mathbf{x}))[W e(\mathbf{g}_\beta(\mathbf{x})) - W e(\mathbf{x}) + (1_Q - \rho(\mathbf{g}(\mathbf{x})))\, W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x})$

If $\mathbf{g}(\mathbf{x}) \in Ee_1$ then $\rho(\mathbf{g}(\mathbf{x}e)) = 0$ and $\rho(\mathbf{x}) = 1$ and (61) simplifies to$_\beta$ =

$W_\beta(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) \leq -2\alpha\|\mathbf{x}\|$.

$W(\mathbf{g}$ $(\mathbf{x})) - W(\mathbf{x}) = V_P(\mathbf{g}(\mathbf{x})) - \sim$ $W(\mathbf{x})$.

Now $\mathbf{g}(\mathbf{x}) \in E_1$ implies $V_P(\mathbf{g}(\mathbf{x})) = \|\mathbf{g}(\mathbf{x})\|_P < r/2$ and since $\mathbf{x} \in E_2 \cap D$, we have $e_\beta\, W(\mathbf{g}(\mathbf{x})) -$ $W(\mathbf{x}) < r/2 -$

$W e_\beta(\mathbf{x}) \leq -2\alpha\|\mathbf{x}\|_Q$ $W(\mathbf{x}) = W(\mathbf{x})$. Thus, by (58)

and we have proved that the second estimate in (49) holds true.

e) By construction $W(\mathbf{x}) = V_P(\mathbf{x}) = \|\mathbf{x}\|_P$ for all $\mathbf{x} \in \mathcal{E}_1$ and $\mathcal{E}_1$ is an open neighbourhood of the origin. Thus, for small enough $\delta > 0$ we have $\mathcal{B}_\delta \subset \mathcal{E}_1$ and (50) follows.

Remark 20. The second order derivatives of W will in general diverge at the origin, but at a predictable rate as stated by (48).

Remark 21. The next theorem, the main result of this paper, is valid for more general sequences $(\mathcal{T}_K)_{K\in\mathbb{N}_0}$ of triangulations, where $\mathcal{T}_{K+1}$ is constructed from $\mathcal{T}_K$ by scaling and tessellating its simplices, than for the sequence $(\mathcal{T}_K)_{K\in\mathbb{N}_0}$ in Definition 3.1. However, it is quite difficult to get hold of the exact conditions that must be fulfilled in a simple way so we restrict the theorem to this specific sequence.

Now we are ready for the main results of this paper.

Theorem 4.2. *Consider the system (1) and assume that the origin is an exponentially stable equilibrium of the system with basin of attraction* $\mathcal{A}$. *Assume that* $\mathcal{D}$ *in Definition 3.1 is a subset of* $\mathcal{A}$. *Then, for every large enough* $K \in \mathbb{N}_0$, *the linear programming problem* L_K *in Definition 3.1 possesses a feasible solution. Especially, the algorithm in the same definition succeeds in computing a CPA Lyapunov function for the system in a finite number of steps.*

Proof. We show that for all large enough $K \in \mathbb{N}_0$ the linear programming problem L_K has a feasible solution. Let us first consider the matrices and constants that are used to initialize the linear programming problem L_K, $K \in \mathbb{N}_0$. The matrices P and Q and then the constants $\lambda^P_{\min}$, $\lambda^P_{\max}$, and $H_{\max}$, are all independent of K. So are the constants B_ν and G_ν because $\mathcal{D} = \mathcal{D}_K = \mathcal{D}_0$ for all $K \in \mathbb{N}_0$. Indeed we set $B_\nu := B$ and $G_\nu := G$ in the algorithm for all $K \in \mathbb{N}_0$, which implies that $G_\mathcal{F}$ is also independent of $K \in \mathbb{N}_0$ (since G is the same for all simplices). In contrast to this, the constants h_ν, $h_{\mathcal{I}\setminus\mathcal{F}}$, $h_{\partial\mathcal{F},P}$ and $E_\mathcal{F}$ do depend on $K \in \mathbb{N}_0$.

For a particular $K \in \mathbb{N}_0$ we have for these constants in the linear programming problem L_K that for an $\mathfrak{S}_\nu \in \mathcal{T}_K = \mathcal{T}_{K,\mathcal{F}^{std}_K}$,

$$h_\nu := \max_{\mathbf{x},\mathbf{y}\in\mathfrak{S}_\nu} \|\mathbf{x}-\mathbf{y}\|_2 = \sqrt{n}\,2^{-2K}F_0 \text{ if } \mathfrak{S}_\nu \subset \mathcal{D}\setminus\mathcal{F}^\circ_K, \tag{66}$$

which implies

$$h_{\mathcal{I}\setminus\mathcal{F}} := \max_{\mathbf{x},\mathbf{y}\in\mathfrak{S}_\nu} \|\mathbf{x}-\mathbf{y}\|_2 = \sqrt{n}\,2^{-2K}F_0,$$

and

$$2^{-K}F_0 = F_K \le h_\nu \le \sqrt{n}F_K = \sqrt{n}\,2^{-K}F_0 \text{ if } \mathfrak{S}_\nu \subset \overline{\mathcal{F}_K}.$$

Similarly

$h_{\partial\mathcal{F},P} := \max\{\|\mathbf{x}-\mathbf{y}\|_P : \mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$ vertices of $S^\nu \subset \mathcal{F}_K\}$

$$\leq \sqrt{\lambda_{\max}^P(n-1)} \cdot 2^{-2K} F_0 \quad (67)$$

and

$$E_{\mathcal{F}} := G_{\mathcal{F}} \cdot \max\left\{H_{\max} \cdot (h_{\mathcal{I}\setminus\mathcal{F}})^2/F_K, 2h_{\partial\mathcal{F},P}\right\}$$
$$\leq 2^{-2K} F_0 G_{\mathcal{F}} \max\left\{H_{\max} n 2^{-K}, 2\sqrt{\lambda_{\max}^P(n-1)}\right\} \quad (68)$$

in $\mathcal{L}_K$.

Set $V_{\mathbf{x}_i} = W(\mathbf{x}_i)$ for all vertices $\mathbf{x}_i$ of all simplices $\mathfrak{S}$ of the triangulation $\mathcal{T}_K$, where W is the function from Lemma 4.1 for the system. Further, set the variable C equal to nC_*, where C_* is the constant from Lemma 4.1. We show that the linear constraints (I)-(IV) in Definition 2.9 are fulfilled for $\mathcal{L}_K$, whenever $K \in \mathbb{N}_0$ is large enough.

For all K so large that $\mathcal{I}_K \subset \mathcal{B}_\delta$, the constraints (I) are fulfilled for $\mathcal{L}_K$ by (50). For all $K \in \mathbb{N}_0$, the constraints (II) for $\mathcal{L}_K$ are fulfilled by (49). By the Mean Value Theorem and (46) we have $|(\nabla V_\nu)_i| \leq C_*$ independent of i and ν and therefore the constraints (III) are fulfilled for $\mathcal{L}_K$. We come to the constraints (IV).

Let $\mathbf{x}_i \neq \mathbf{0}$ be an arbitrary vertex of an arbitrary simplex $\mathfrak{S}_\nu \in \mathcal{T}_K$, $\mathfrak{S}_\nu \subset \mathcal{O}_K = \mathcal{O}_0$. Then $\mathbf{g}(\mathbf{x}_i) \in \mathfrak{S}_\mu$ for some simplex $\mathfrak{S}_\mu = \mathrm{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n) \in \mathcal{T}_K$ and we have $\mathbf{g}(\mathbf{x}_i) = \sum_{j=0}^n \mu_j \mathbf{y}_j$. We have assigned $V_{\mathbf{x}} = W(\mathbf{x})$ for all vertices $\mathbf{x}$ of all simplices $\mathfrak{S}$ of the triangulation $\mathcal{T}_K$. Hence,

$$\sum_{j=0}^n \mu_j V_{\mathbf{y}_j} - V_{\mathbf{x}_i} = \sum_{j=0}^n \mu_j W(\mathbf{y}_j) - W(\mathbf{x}_i)$$

$$= \sum_{j=0}^n \mu_j W(\mathbf{y}_j) - W\left(\sum_{j=0}^n \mu_j \mathbf{y}_j\right) + \underbrace{W(\mathbf{g}(\mathbf{x}_i)) - W(\mathbf{x}_i)}_{\leq -2\alpha\|\mathbf{x}_i\|_Q \text{ by } (49)}$$

If $\mathfrak{S}_\mu \subset \mathcal{D} \setminus \mathcal{F}_K^\circ$, then we can use Proposition 2.10, (48) with $\epsilon = F_K$ and (66) to get the estimate

$$\left|\sum_{j=0}^n \mu_j W(\mathbf{y}_j) - W\left(\sum_{j=0}^n \mu_j \mathbf{y}_j\right)\right| \leq nA_\epsilon h_\nu^2 \leq n\frac{A}{F_K} h_\nu^2 = n^2 A F_0 2^{-3K}. \quad (69)$$

Thus,

$$\sum_{j=0}^n \mu_j V_{\mathbf{y}_j} - V_{\mathbf{x}_i} + CG_\nu h_\nu \leq n^2 A F_0 2^{-3K+1} - 2\alpha\|\mathbf{x}_i\|_Q + CG\sqrt{n}\, 2^{-2K} F_0$$

and the are fulfilled $n^2AF_02^{-3K} - 2\alpha\|\mathbf{x}_i\|_Q + CG\sqrt{n}\,2^{-2K}F_0 \leq -\alpha\|\mathbf{x}_i\|_Q$ constrains (21) if .

Because $\alpha\|\mathbf{x}_i\|_Q \geq \alpha F_K\sqrt{\lambda_{\min}^Q} = \alpha 2^{-K}F_0\sqrt{\lambda_{\min}^Q}$

and holds true

$$n^2AF_02^{-3K} + CG\sqrt{n}\,2^{-2K}F_0 \leq \alpha 2^{-K}F_0\sqrt{\lambda_{\min}^Q}$$

for all large enough $K \in \mathrm{N}_0$, we get√ ≤ || ||

$n^2AF_02^{-3K} + CG$ $n2^{-2K}F_0$ α $\mathbf{x}_{i\,Q}$ (70)

and the constrains (21) are fulfilled for all large enough $K \in \mathrm{N}_0$.

If $\mathrm{S}_\mu \subset \mathrm{F}_K$ and K is so large that $\mathrm{F}_K \subset \mathrm{B}_\delta$, then we have $W(\mathbf{y}_j) = \|\mathbf{y}_j\|_P$ for $j = 0,1,\dots,n$ and we can use the estimate (40) in the proof of Theorem 2.11 to get the estimate

$$\sum_{j=0}^{n}\mu_jW(\mathbf{y}_j) - W\left(\sum_{j=0}^{n}\mu_j\mathbf{y}_j\right) \leq 2h_{\partial\mathcal{F},P} \leq F_02^{-2K+1}\sqrt{\lambda_{\max}^P(n-1)}, \tag{71}$$

using (67). Thus, by (68) and $h_\nu\|\mathbf{x}_i\|_2 \leq \sqrt{n}F_K$ we have Since

$$\begin{aligned}\sum_{j=0}^{n}\mu_jV_{\mathbf{y}_j} - V_{\mathbf{x}_i} + B_\nu Ch_\nu\|\mathbf{x}_i\|_2 + E_{\mathcal{F}} &\\ \leq -2\alpha\|\mathbf{x}_i\|_Q + F_02^{-2K+1}\sqrt{\lambda_{\max}^P(n-1)} + BCn2^{-2K}F_0^2 &\\ +2^{-2K}F_0G_{\mathcal{F}}\max\left\{H_{\max}n2^{-K}, 2\sqrt{\lambda_{\max}^P(n-1)}\right\}.&\end{aligned}$$

$\|\mathbf{x}_i\|_Q \geq F_K\sqrt{\lambda_{\min}^Q} = 2^{-K}F_0\sqrt{\lambda_{\min}^Q}$ we get, similarly to (70), that the constraints (22) are fulfilled if

$$F_02^{-2K+1}\sqrt{\lambda_{\max}^P(n-1)} + BCn2^{-2K}F_0^2 + 2^{-2K}F_0G_{\mathcal{F}}\max\left\{H_{\max}n2^{-K}, 2\sqrt{\lambda_{\max}^P(n-1)}\right\}$$
$$\leq \alpha 2^{-K}F_0\sqrt{\lambda_{\min}^Q},$$

which again is clearly the case for all large enough K.

5. Example

As a proof of concept, we compute a CPA Lyapunov function by the methods described in this paper as an example. We consider the system

$$x_{k+1} = \frac{1}{2}x_k + x_k^2 - y_k^2, \qquad y_{k+1} = -\frac{1}{2}y_k + x_k^2 \tag{72}$$

from [11]. That is, the system (1) with

$\mathbf{g}(\mathbf{x}) = \mathbf{g}(x,y) = \begin{pmatrix} 0.5x + x^2 - y^2 \\ -0.5y + x^2 \end{pmatrix}.$

With

$x^\nu := \max_{(x,y)\in\mathfrak{S}_\nu} |x|$, $\quad y^\nu := \max_{(x,y)\in\mathfrak{S}_\nu} |y|$

we can assign

$G_\nu := 2 \cdot \max\{0.5 + 2x^\nu, 2y^\nu\}$ and $B_\nu := 2 \cdot 2 = 4$

for all $S_\nu \in T$ in the linear programming problem from Definition 2.9. The Jacobian matrix of **g** at the origin is given by

$$A := D\mathbf{g}(\mathbf{0}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

We set $Q := I$, i.e. the identity matrix, which results in $P := 4/3 \cdot I$ being the solution to the discrete Lyapunov equation (8). We take

$$\max_{S_\nu \in \mathcal{T}} \|\nabla V_\nu\|_\infty$$

as the objective function of our linear programming problem, and we minimize it. Thus, a feasible solution with a flat gradient will be selected in order to obtain equally distributed level sets of the Lyapunov function.

We solve the linear programming problem from Definition 2.9, constructed for the system (72) with the triangulation $T_{K,F}^{\text{std}}$, where the parameters are $K = 4$, $F := 0.033$, $N_I := 2$, $N_O := 10$, and $N_D := 12$. For these parameters the linear programming problem has a feasible solution, which was computed using the Gnu Linear Programming Kit (http://www.gnu.org/software/glpk/) from Andrew Makhorin. The computed CPA Lyapunov function is depicted in Figure 4. As described in Definition 2.5, a simplicial fan is used to triangulate $F = [-0.033, 0.033]^2$. This simplicial fan is depicted in Figure 5. The domain of the computed CPA Lyapunov function is $D = [-N_D \cdot F, N_D \cdot F]^2 = [-0.396, 0.396]^2$. The largest connected component of a sublevel set compact in $O = [-N_I \cdot F, N_I \cdot F]^2 = [-0.33, 0.33]^2$ is assured to be in the basin of attraction of the equilibrium at the origin, cf. Remark 3. This set is depicted in Figure 3.

Let us compare these result with the quadratic Lyapunov function, obtained by solving the discrete Lyapunov equation. By equation (51), $\widetilde{V}_P(\mathbf{x}) := \mathbf{x}^T P \mathbf{x}$, $\widetilde{V}_P(\mathbf{g}(\mathbf{x})) - \widetilde{V}_P(\mathbf{x}) < 0$ for all $\mathbf{x}$ such that

$$\|\boldsymbol{\psi}(\mathbf{x})\|_2 \|P\|_2 \left(\|\boldsymbol{\psi}(\mathbf{x})\|_2 + 2\|A\|_2\right) = \frac{4}{3} \cdot \|\boldsymbol{\psi}(\mathbf{x})\|_2 \left(\|\boldsymbol{\psi}(\mathbf{x})\|_2 + 1\right) < 1$$

where $\psi(\mathbf{x}) = (\mathbf{g}(\mathbf{x}) - A\mathbf{x})/\|\mathbf{x}\|_2$; note that $\|\mathbf{x}\|_2 = \|\mathbf{x}\|_Q$. By using the general estimate derived directly above inequality (4.6) in [17], we get for all $\|\mathbf{x}\|_2 = r > 0$ that is a function system

$$\|\boldsymbol{\psi}(\mathbf{x})\|_2 \le \frac{r^2}{2\|\mathbf{x}\|_2}\sqrt{\sum_{i=1}^{2}\left(\sum_{k,j=1}^{2}\max_{\boldsymbol{\xi}\in[-r,r]^2}\left|\frac{\partial^2 g_i}{\partial x_j \partial x_k}(\boldsymbol{\xi})\right|\right)^2} = r$$

Lyapunov $\sqrt{5}$ for the in the set

$\{\mathbf{x} : \|\mathbf{x}\|_2 < \sqrt{5}/10\} \approx$ 0.224.

In Figure 3 we compare these lower bounds on the basin of attraction with the lower bounds delivered by the CPA Lyapunov function from above. For further comparison we solved the linear programming problem from Definition 2.9 for the same system with the parameters $K = 5$, $F = 0.1$, $N_I = 2$, $N_O = 4$, and $N_D = 6$. Moreover, we exclude (22) with F = $[-0.1,0.1]^2$ from the constraints (IV). Note, that in this case the sublevel set in Figure 3 is a forward invariant set with the property that

and $\widetilde{V}_P$ $\mathcal{B}$

for anythat $\phi(t\xi_k$,in the sublevel set, there exists a sequence $(\xi) \in$ F = $[-0.1,0.1]^2$ for all $k \in$ N. Since Ft^k= $[)^{k\in}$−N0.with1,0.1]t^k2 is a subset of→ +∞, such

the basin of attraction, as shown by the quadratic Lyapunov function, we can also conclude that the sublevel set is a subset of the basin of attraction.

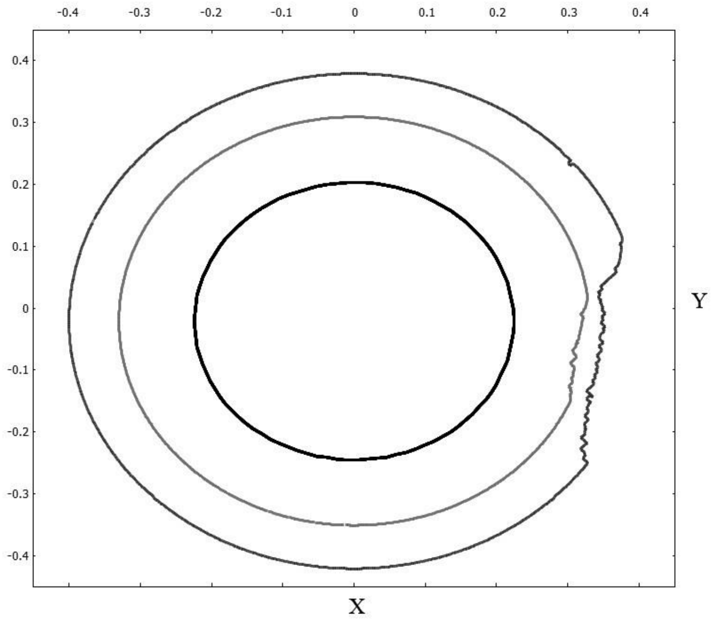


Figure 3. The figure shows three subsets of the basin of attraction. The smallest one is obtained by the quadratic Lyapunov function, derived from the discrete Lyapunov equation, the middle one is obtained by the CPA Lyapunov function with the simplicial fan at the origin, and the largest one is obtained by the CPA Lyapunov function excluding the set F = $[-0.1,0.1]^2$.

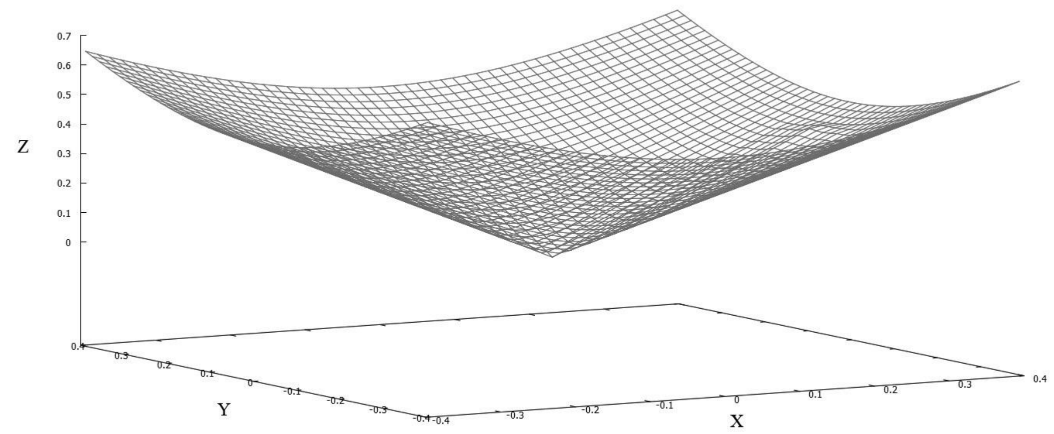
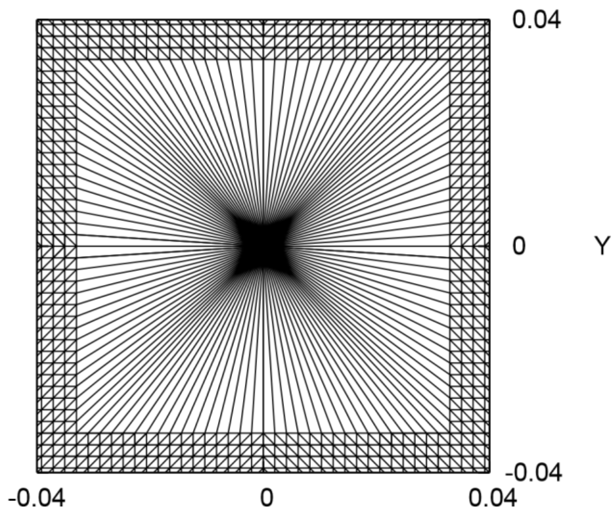


Figure 4. The CPA Lyapunov function without the fan computed for the system (72). The CPA Lyapunov function computed with the fan looks very similar but is defined on a smaller domain.



X
Figure 5. The simplicial fan and its closest neighbourhood of the simplicial complex. $\mathcal{T}^{\mathrm{std}}_{4,0.033}$

6. Conclusion and Future Directions

In this paper, we fully adapted the CPA method to compute Lyapunov functions to autonomous discrete systems. In Definition 2.9 we presented a linear programming problem, of which a feasible solution parameterizes a CPA Lyapunov function for the system in question. In Definition 3.1 we offered an algorithm that generates linear programming problems as in Definition 2.9 for ever more refined triangulations of a hypercube D containing the origin. In Theorem 4.2 we proved, that if the system at hand has an exponentially stable equilibrium at the origin and D is a subset of *REFERENCES* its region of attraction, then the algorithm succeeds in a finite number of steps in computing a CPA Lyapunov function for the system. Finally, in Section 5, we have applied the method to an example and have computed a CPA Lyapunov function.

The CPA method for continuous systems has been extended to compute CPA Lyapunov functions for switched systems [19] and differential inclusions [2, 3]. It seems very promising for further research in this direction to combine the theory on the stability of difference inclusions and smooth Lyapunov functions given in [24–27] with the theory developed in this paper to design an algorithm to compute CPA Lyapunov functions for exponentially stable difference inclusions.

References

[1] R. Agarwal, "Difference Equations and Inequalities: Theory, Methods, and Applications", 2. ed., Marcal Dekker, Inc., 2000.

[2] R. Baier, L. Gru¨ne, and S. Hafstein, Computing Lyapunov functions for strongly asymptotically stable differential inclusions. In: Proceedings of NOLCOS (2010).

[3] R. Baier, L. Gru¨ne and S. Hafstein, Linear programming based Lyapunov function computation for differential inclusions, Discrete Contin. Dyn. Syst. Ser. B, **17** (2012), 33–56.

[4] H. Ban and W. Kalies, A computational approach to Conley's decomposition theorem, J. Comput. Nonlinear Dynam., **1-4** (2006), 312–319.

[5] A. Barraud, 'An Algorithm for Solving the Matrix Equation $X = F X F^T + S$'. Int. J. Control, **27-2** (1978), 319–320.

[6] R. Bartels and G. Stewart, Algorithm 432: Solution of the matrix equation $AX+XB = C$, Comm. ACM, **15** (1972), 820–826.

[7] D. Bertsimas and J. Tsitsiklis, "Introduction to linear optimization", Athena Scientific, 1997.

[8] M. Buhmann, "Radial basis functions: theory and implementations", Cambride University Press, 2003.

[9] C. Conley "Isolated invariant sets and the Morse index", Vol. 38 of CBMS, American Math. Soc., 1978.

[10] G. Feng, Stability Analysis of Piecewise Discrete-Time Linear Systems IEEE Trans. Automatic Control **47-7** (2002), 1108–1112.

[11] P. Giesl, On the determination of the basin of attraction of discrete dynamical systems. J. Difference Equ. Appl., **13-6** (2007), 523–546.

[12] P. Giesl, Construction of a local and global Lyapunov function for discrete dynamical systems using radial basis functions. Journal of Approximation Theory, **153-2** (2008), 184–211.

[13] P. Giesl and S. Hafstein, Existence of piecewise affine Lyapunov functions in two dimensions. J. Math. Anal. Appl., **371** (2010), 233–248.

[14] P. Giesl and S. Hafstein, Construction of Lyapunov functions for nonlinear planar systems by linear programming, J. Math. Anal. Appl., **388** (2012), 463–479.

[15] P. Giesl and S. Hafstein, Existence of piecewise linear Lyapunov functions in arbitary dimensions, Discrete Contin. Dyn. Syst. Ser. A, **32** (2012), 3539–3565.

[16] P. Giesl and S. Hafstein, Revised CPA method to compute Lyapunov functions for nonlinear systems, (submitted 2013).

[17] S. Hafstein, A constructive converse Lyapunov theorem on exponential stability. Discrete Contin. Dyn. Syst. Ser. A, **10** (2004), 657–678.

[18] S. Hafstein, A constructive converse Lyapunov theorem on asymptotic stability for nonlinear autonomous ordinary differential equations. Dynamical Systems, **20** (2005), pp. 281–299

[19] S. Hafstein, "An algorithm for constructing Lyapunov functions", Electron. J. Differential Equ., Monogr. **8** (2007).

REFERENCES

[20] Z.-P. Jiang and Y. Wang, A converse Lyapunov theorem for discrete-time systems with disturbances. System & Control Letters, **45** (2002), 49–58.

[21] P. Julian, "A high level canonical piecewise linear representation: Theory and applications", Ph.D. Thesis: Universidad Nacional del Sur, Bahia Blanca, Argentina, 1999.

[22] P. Julian, J. Guivant and A. Desages, A parametrization of piecewise linear Lyapunov function via linear programming, Int. Journal of Control, **72** (1999), 702–715.

[23] W. Kalies, K. Mischaikow and R. VanderVorst, An algorithmic approach to chain recurrence, Found. Comput. Math., **5-4** (2005), 409–449.

[24] C. Kellet, "Advances in Converse and Control Lyapunov Functions", Ph.D. Thesis: University of California, Santa Barbara, USA, 2002.

[25] C. Kellet and A. Teel, Results on Converse Lyapunov Theorems for Difference Inclusions, In: Proceedings of the 42st IEEE Conference on Decision and Control, **4** (2003), 3627–3632.

[26] C. Kellet and A. Teel, Smooth Lyapunov Functions and Robustness of Stability for Difference Inclusions, System Control Lett., **52-5** (2005), 395–405.

[27] C. Kellet and A. Teel, On the Robustness of KL-Stability for Difference Inclusions: Smooth Discrete-Time Lyapunov Functions, SIAM J. Control Optim., **44-3** (2005), 777–800.

[28] H. Khalil, "Nonlinear Systems", Prentice Hall, 3. ed., 2002.

[29] G. Kitagawa, An Algorithm for Solving the Matrix Equation X = F X F' + S. Int. J. Control, **25-5** (1977), 745-753.

[30] M. Lazar and A. Jokic, On infinity norms as Lyapunov functions for piecewise affine systems. In: Proceedings of Hybrid Systems: Computation and Control conference (2010), 131–141.

[31] A. Lyapunov, "The general problem of the stability of motion", translated by A. T. Fuller, Taylor & Francis, 1992.

[32] S. Marinosson, "Stability analysis of nonlinear systems with linear programming: A Lyapunov functions based approach", Ph.D. Thesis: Gerhard-Mercator-University, Duisburg, Germany, 2002.

[33] S. Marinosson, Lyapunov function construction for ordinary differential equations with linear programming. Dynamical Systems, **17** (2002), 137–150.

[34] B. Milani, Piecewise-affine Lyapunov functions for discrete-time linear systems with saturating controls. Automatica, **38** (2002), 2177–2184.

[35] D. Norton, The fundamental theorem of dynamical systems, Comment. Math. Univ. Carolinae, **36-3** (1995), 585–597.

[36] C. Sanderson, "Armadillo: An Open Source C++ Linear Algebra Library for Fast Prototyping and Computationally Intensive Experiments", Technical Report, NICTA, 2010.

[37] S. Sastry, "Nonlinear Systems: Analysis, Stability, and Control.", Springer, 1999.

[38] E. Sontag, "Mathematical Control Theory: Deterministic Finite Dimensional Systems.", 2. ed., Springer, 1998.

[39] L. Tu, "An Introduction to Manifolds", Springer, 2008.

[40] H. Wendland, "Scattered Data Approximation.", Cambride University Press, 2004.

[41] V. Zubov, "Methods of A.M.Lyapunov and their application.", P.Noordhoff, Groningen, Netherlands, 1964.